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UNIFYING THE ANALYTICAL SOLUTIONS OF THE GEODESIC EQUATIONS IN BONDI-GOLD-HOYLE UNIVERSE MODEL

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Abstract. In this paper, we unify the study of the geodesics in the Bondi-Gold-Hoyle universe model. Thus, we obtain analytical solutions of the geodesic equations which describe the motion of space-like, time-like and light-like particles in the considered model. In the case of light-like geodesics, we get the solutions obtained by Marcheva & Ivanov, 2017.

Keywords: universe models; steady state theory; differential geometry; differential equations; geodesic equations

Introduction

Investigating the nature of the universe and the motion of the particles are among the most important problems in natural sciences and physics education. These problems are closely related to the general relativity, i.e., to the geometric theory of gravitation. It is well known that the different solutions of the Einstein field equations, yield different metrics of the spacetime. Depending on the metric, there are a lot of universe models presented in the literature (see e.g., Carroll, 2004); Weinberg, 2008 and references therein). Nevertheless, cosmology is dominated by two alternative paradigms. The first one is called *Big Bang model* and the second one is called *Steady-State theory* (Aguirre & Gratton, 2002). The steady-state theory was first proposed by Einstein in 1931 (in unpublished manuscript). Seventeen years later, steady-state models of the expanding universe were independently proposed by Bondi & Gold (1948) and Hoyle (1948). Recall that, the Bondi-Gold-Hoyle universe model can be defined in a four-dimensional Lorentzian manifold (M^4, g) by the following de Sitter sphere:

$$S_1^4 : \left| \begin{array}{l} z^1 = r \sinh\dfrac{u^1}{r} + \dfrac{e^{u^1/r}}{2r} |\mathrm{u}|^2; \\ z^2 = r \cosh\dfrac{u^1}{r} - \dfrac{e^{u^1/r}}{2r} |\mathrm{u}|^2; \\ z^3 = e^{u^1/r} u^2; \\ z^4 = e^{u^1/r} u^3;\ z^5 = e^{u^1/r} u^4, \end{array} \right. \qquad (1)$$

where r is the radii of S_1^4; $|\mathrm{u}|^2 = (u^2)^2 + (u^3)^2 + (u^4)^2$. For a more detailed historical survey of the Bondi-Gold-Hoyle model, we refer the reader to the elegant paper of O'Raifeartaigh & Mitton (2015).

Let (M^4, g) be a Lorentzian manifold, then the curve $\gamma : J \subset \mathbb{R} \to M^4$ defined by $\gamma : u^k = u^k(s)$ is called *geodesic* if it satisfies the following equations (see e.g., Weinberg (1972; Busemann, 2005; Toponogov, 2006):

$$\frac{d^2 u^k}{ds^2} + \Gamma_{ij}^k \frac{du^i}{ds}\frac{du^j}{ds} = 0 \quad (i, j, k = 1, \ldots, 4), \qquad (2)$$

where Γ_{ij}^k are the Christoffel symbols defined by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) \qquad (3)$$

and (g^{kl}) is the inverse of the matrix (g_{kl}). In the light of the general relativity, the freely moving particles in a curved spacetime always move along a geodesic (Grøn & Næss, 2002). Therefore, it is not surprising that one of the most interesting problems in mathematical physics is solving the geodesic Eq. (2). It is well known that in Cartesian coordinates the Christoffel symbols, Eq. (3), are zeros for all i, j, k and therefore Eq. (2) become equations of straight line. An interesting interactive tool which can be used in the physics education for studying the geodesics in the different universe models has been presented and studied by Müller & Grave (2010) and Müller & Frauendiener (2011).

Let $p \in \gamma$, then the tangent vector to γ in the point p is defined by

$$\dot{\gamma}_p = \left(\frac{du^1}{ds}, \frac{du^2}{ds}, \frac{du^3}{ds}, \frac{du^4}{ds} \right).$$

So, we have

$$\dot{\gamma}_p^2 = g_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \tag{4}$$

in terms of Einstein summation convention. Note that, if $\dot{\gamma}_p$ is a nonzero vector, then the curve γ is called *time-like* in the point p if $\dot{\gamma}_p^2 < 0$, *light-like* or *null* if $\dot{\gamma}_p^2 = 0$ and *space-like* if $\dot{\gamma}_p^2 > 0$ (Yilmaz & Turgut, 2008; İlarslan & Nešović, 2009 and references therein).

Recently, Marcheva & Ivanov (2017) have obtained an analytical solution of the geodesic Eqs. (2) in Bondi-Gold-Hoyle universe model. Eq. (1), in the case of light-like geodesics. In this paper, using a unified approach, we obtain analytical solutions of the geodesic Eqs. (2) which describe the motion of space-like, light-like and time-like particles in Bondi-Gold-Hoyle model and complement the result obtained by Marcheva & Ivanov (2017).

Exact analytical solutions

Since the metric tensor of the model, Eq. (1), has nonzero components

$$g_{11} = -1,\ g_{22} = g_{33} = g_{44} = e^{2u^1/r} \tag{5}$$

then from Eq. (3), we obtain the following nonzero Christoffel symbols

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{e^{2u^1/r}}{r}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \Gamma^4_{14} = \Gamma^4_{41} = \frac{1}{r}. \tag{6}$$

Furthermore, for all $i \in \mathbb{N}$ we use the notation $\dot{u}^i = du^i / ds$ and consider the set of real constants $\{c_i\}$. Hence, setting $u^1 = t, u^2 = x, u^3 = y$ and $u^4 = z$ from Eqs. (2) and (6), we get the following system of equations:

$$\left|\begin{array}{l} \ddot{t} + \dfrac{e^{2t/r}}{r}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = 0; \\ \ddot{x} + \dfrac{2}{r}\,\dot{t}\,\dot{x} = 0;\ \ddot{y} + \dfrac{2}{r}\,\dot{t}\,\dot{y} = 0;\ \ddot{z} + \dfrac{2}{r}\,\dot{t}\,\dot{z} = 0. \end{array}\right. \qquad (7)$$

From the last third equations of Eq. (7), we get $\dot{y} = c_1\,\dot{x}$ and $\dot{z} = c_2\,\dot{x}$ wherefrom, we obtain the following relations

$$y = c_1\,x + c_3,\ z = c_2\,x + c_4 \qquad (8)$$

and

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = A\,\dot{x}^2, \qquad (9)$$

where $A = 1 + c_1^2 + c_2^2$. Taking into account (9), the system, Eq. (7), reduces to the following one

$$\left|\begin{array}{l} \ddot{t} + \dfrac{A\,e^{2t/r}}{r}\,\dot{x}^2 = 0; \\ \ddot{x} + \dfrac{2}{r}\,\dot{t}\,\dot{x} = 0. \end{array}\right. \qquad (10)$$

Let $\in \mathbb{R} \setminus \{0\}$ be fixed. Dividing both sides of the second equality by $\dot{x}$ and integrating with respect to s, we get

$$\dot{x} = c_5\,e^{-2t/r}. \qquad (11)$$

From Eq. (11) and the first equation of (10), we obtain the equation

$$\ddot{t} + \frac{A\,c_5^2}{r}\,e^{-2t/r} = 0.$$

Reducing the order of the last equation, we get

$$-\dot{t}^2 + A\,c_5^2\,e^{-2t/r} = \alpha, \quad \alpha \in \mathbb{R}. \qquad (12)$$

It is important to note that, according to (4), from Eqs. (5), (9), (11) and (12) it follows that

$$\dot{\gamma}_p^2 = -\dot{t}^2 + e^{2t/r}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = -\dot{t}^2 + A\,c_5^2\,e^{-2t/r} = \alpha.$$

This identity shows that the choice of the constant α determines whether the geodesic γ is time-like, light-like or space-like.

In what follows, replacing successively $\dot{x}$ from Eq. (11) and $A\,c_5^2\,e^{-2t/r}$ from Eq. (12) in the first equation of (10), we obtain the following system of equations:

$$\left|\begin{array}{l} \ddot{t}+\dfrac{\dot{t}^2+\alpha}{r}=0; \\ \ddot{x}+\dfrac{2}{r}\,\dot{t}\,\dot{x}=0. \end{array}\right. \tag{13}$$

In the case $\alpha = 0$, i.e., in the case of lightlike geodesic the solutions of (13) coincide with the solutions obtained by Marcheva & Ivanov (2017, Eq. (7)). Let $\alpha \neq 0$, then a solution of the first equation of (13) is the function

$$t(s) = r\ln\left(\frac{c_6^{\frac{\sqrt{-\alpha}}{r}}}{2\sqrt{-\alpha}}\left(e^{\pm\frac{\sqrt{-\alpha}}{r}s} - B\,e^{\mp\frac{\sqrt{-\alpha}}{r}s}\right)\right), \tag{14}$$

where $B = Ac_5^2/(c_6^{2\sqrt{-\alpha}/r})$ whenever $c_6 \neq 0$ and $e^{2t/r} \neq Ac_5^2/\alpha$. Substituting t from (14) into (11), we get the equation

$$\dot{x} = -\frac{4\alpha\, c_5}{c_6^{2\sqrt{-\alpha}/r}}\left(e^{\pm\frac{\sqrt{-\alpha}}{r}s} - B\,e^{\mp\frac{\sqrt{-\alpha}}{r}s}\right)^{-2}$$

whose solution is

$$x(s) = \pm\frac{2rc_5\sqrt{-\alpha}}{c_6^{2\sqrt{-\alpha}/r}}\left(B - e^{\pm\frac{\sqrt{-\alpha}}{r}s}\right)^{-1}. \tag{15}$$

Finally, from (15) and (8), we obtain the following solutions for y and z:

$$y(s) = \pm\frac{2rc_1c_5\sqrt{-\alpha}}{c_6^{2\sqrt{-\alpha}/r}}\left(B - e^{\pm\frac{\sqrt{-\alpha}}{r}s}\right)^{-1} + c_3 \tag{16}$$

and

$$z(s) = \pm\frac{2rc_2c_5\sqrt{-\alpha}}{c_6^{2\sqrt{-\alpha}/r}}\left(B - e^{\pm\frac{\sqrt{-\alpha}}{r}s}\right)^{-1} + c_4. \tag{17}$$

Conclusions

In this paper, we have obtained analytical solutions of the geodesic equations in the Bondi-Gold-Hoyle universe model. Reducing the order of the first equation of the main system, Eq, (7), we have extracted a constant α which determines whether the geodesic γ is time-like, light-like or space-like. More precisely, for $\alpha < 0$ the solutions, Eqs. (14) – (17), describe the paths of time-like particles in the considered model, for $\alpha = 0$ – the paths of massless particles and for $\alpha > 0$ – the paths of massive particles.

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