UBIQUITOUS ARCHIMEDEAN CIRCLES

Hiroshi Okumura

Abstract. Ubiquitous Archimedean circles of the arbelos arising from a result of the old Japanese geometry are considered. It is shown that they cover the plane.
Keywords: arbelos, Archimedean circle, paralleogram, similarity, ubiquitous circle.

1. Introduction. For a point C on the segment AB, let us consider three semicircles α, β and γ with diameters AC, BC and AB, respectively constructed on the same side. The area surrounded by the three semicircles is called an arbelos. Let I be the point of intersection of the perpendicular to the line AB passing through the point C and the semicircle γ. The line IC divides the arbelos into two curvilinear triangles with congruent incircles, which are called the twin circles of Archimedes. Circles congruent to the twin circles are called Archimedean circles of the arbelos. Let a and b be the radii of the semicircles α and β respectively. The radii of the Archimedean circles are expressed by:

$$(1) \qquad \frac{ab}{a+b}.$$

In (Dodge et al., 1999) many Archimedean circles are demonstrated, which are likened to being "ubiquitous" as in the title of the paper. But their existence is restricted in a certain narrow area in the plane. We consider circles, which should at least cover the plane if they are ubiquitous.

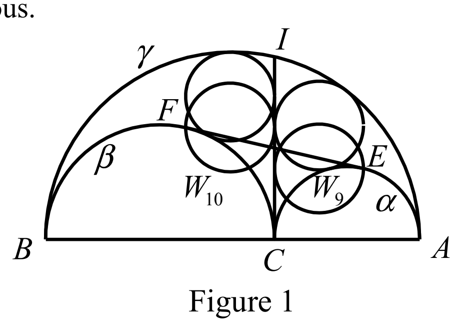

Figure 1

Let us assume that the external common tangent of α and β touches the two circles at points E and F respectively. Then each of the two minimal circles passing through the points E and F respectively and touching the line IC is Archimedean. The minimal circles are denoted by W_9 and W_{10} in (Dodge et al., 1999) (see Figure 1). In this article we show the existence of Archimedean circles covering the plane by generalizing those Archimedean circles by a simple result of the old Japanese geometry.

2. New Archimedean twin circles. We use the following lemma in the Japanese geometry (Gazen Yamamoto, 1841) (The original text states the case in which *PS*, *QR* and *TU* are perpendicular to *RS*.) (see Figure 2)

Lemma. *For a point U on the segment RS, let PS, QR and TU be parallel segments erected on the same side of RS such that T is the point of intersection of the segment PR and QS. Then*

$$\frac{1}{|PS|}+\frac{1}{|QR|}=\frac{1}{|TU|}.$$

Proof: Let $r=|RU|$ and $s=|SU|$. By the similar triangles we have

$\frac{r+s}{|PS|}=\frac{r}{|TU|}$ and $\frac{r+s}{|QR|}=\frac{s}{|TU|}$. This implies $\frac{r+s}{|PS|}+\frac{r+s}{|QR|}=\frac{r+s}{|TU|}$. QED

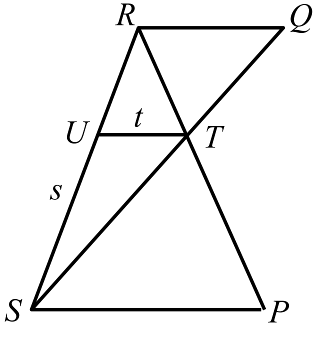

Figure 2

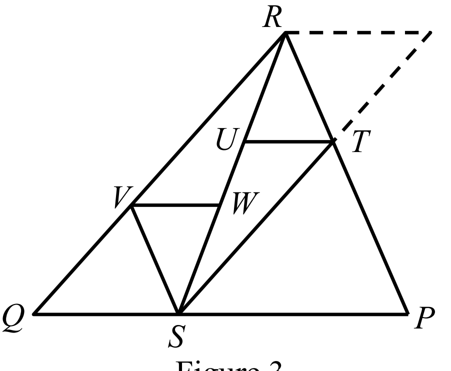

Figure 3

For a triangle *PRQ* with a point *S* on the segment *PQ*, let *T* and *V* be points on the segments *PR* and *QR* respectively such that *STRV* is a parallelogram (see Figure 3). If *U* and *W* are points on the segment *SR* such that *TU* and *VW* are parallel to *PQ*, then by the Lemma we have

$$\frac{1}{|TU|}=\frac{1}{|VW|}=\frac{1}{|PS|}+\frac{1}{|QS|}.$$

From this observation with (1) we get the following theorem (see Figure 4):

Theorem. *For a point D, which does not lie on the line AB, let E and F be points on the segment AD and BD respectively, such that CEDF is a parallelogram. If G and H are points on the segment CD such that EG and FH are parallel to AB, then the circles with diameters EG and FH are Archimedean.*

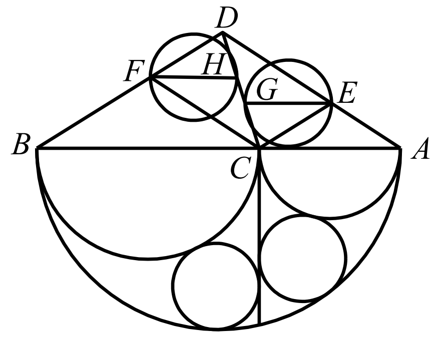

Figure 4

3. Parallelograms and ubiquitous Archimedean circles. Since the three semicircles α, β and γ are not involved in the theorem the reader might think that the Archimedean circles obtained by the theorem are not so closely related to the arbelos. But we can show that for a point D, not lying on the line AB, the parallelogram $CEDF$ in the theorem is constructed by the arbelos (see Figure 5): To avoid the case that there is no point of intersection, we now assume that α, β and γ are circles. Let I and J be the points of intersection of the circle γ and the lines AD and BD respectively, where the line AJ intersects α at a point K and the line BI intersects β at a point L. Let E be the point of intersection of the lines CK and AD and let F be the point of intersection of the lines CL and BD. Then $CEDF$ is a parallelogram.

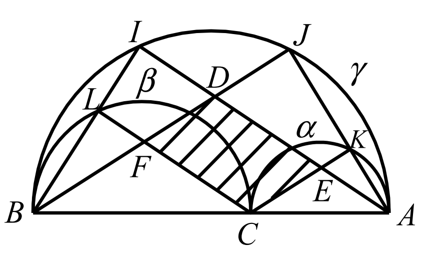

Figure 5

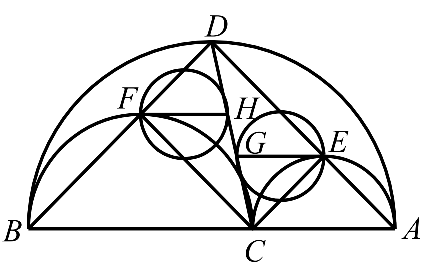

Figure 6

Let us assume that the point D lies on γ. Then the points E and F also lie on the circles α and β respectively and $CEDF$ is a rectangle (see Figure 6). Since the triangles DBA, FBC and ECA are all similar, if D is the furthest point on γ from AB, so do E and F on α and β, respectively. Indeed Figure 6 shows this case. If CD is perpendicular to AB, then E and F coincide with the points of tangency of α and β with one of the external common tangents of the two (Bankoff, 1994). This is the case indicated in Figure 1.

For a point E, not lying on the line AB, there are points D and F such that the points D, E and F satisfy the condition of the theorem. This implies that for any point, not lying on the line AB, there is an Archimedean circle whose one of the endpoints of the diameter parallel to AB is just the point. Therefore our Archimedean circles cover the plane.

The parallelism and the similarity are closely related concepts. Therefore it is not unexpected to see a parallelogram in the topics of the arbelos, since it consists of three mutually similar figures. This fact makes possible the use of the presented properties in secondary school teaching which will be considered in another publication. Examples of such teaching (Grozdev & Watanabe, 2011) apply the theory of the arbelos in the formation of mathematical knowledge based on the developing education principle. Also, for a Java applet, which indicates the Archimedean circles in the present paper, see:

http://www.retas.de/thomas/arbelos/okumura/ubiquitous.html

REFERENCES

1. Bankoff, L. (1994). The marvelous arbelos (pp. 247–253). In: Guy, R. K. & Woodrow, R. E. (Eds.). *The lighter side of Mathematics*. Mathematical Association of America.
2. Dodge, C. W., Schoch, T., Woo, P. Y. & Yiu, P. (1999). Those ubiquitous Archimedean circles. *Mathematics Magazine*, 72, 202–213.
3. Gazen Yamamoto (1841). *Samp¯o Jojutsu*.
4. Grozdev, S., Watanabe, M. (2011). The generalized arbelos as an example of instrumentarium for developing education in Japan (380–386). In: *Mathematics and Mathematical Eduction, Proceedings of the 40th Spring Conference of UBM, Borovets, April 5 – 9, 2011*. Sofia: UBM.

ПОКРИВАЩИ РАВНИНАТА АРХИМЕДОВИ ОКРЪЖНОСТИ

Хироши Окумура

Резюме. Статията е посветена на един клас Архимедови окръжности на арбелоса (обущарски нож), който се появява въз основа на резултат от старояпонската геометрия. Показано е, че окръжностите от този клас покриват равнината.

✉ **Hiroshi Okumura**
Professor
251 Moo 15 Ban Kesorn, Tambol Sila
Amphur Muang Khonkaen 40000
Thailand
hiroshiokmr@gmail.com