

THREE SOLUTIONS OF A PROBLEM WITH FOUR CIRCLES

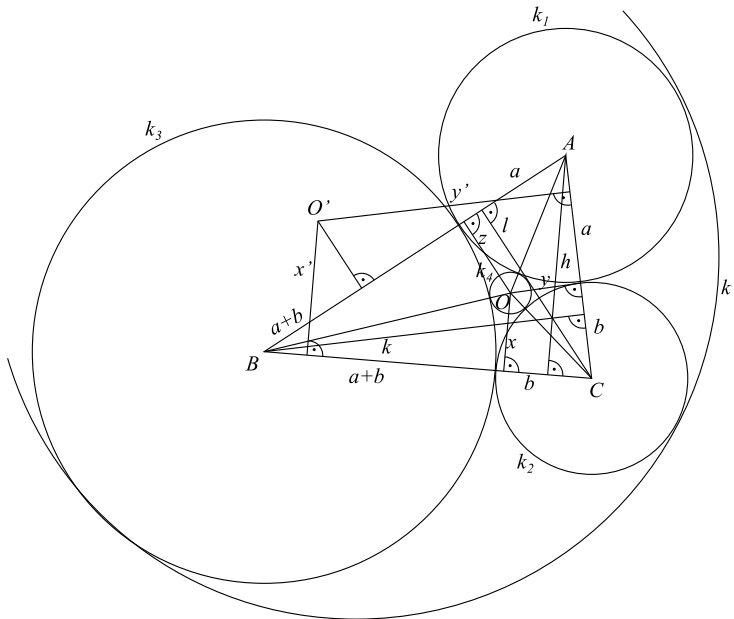
Šefket Arslanagić
University of Sarajevo

Abstract. The paper considers three solutions of an interesting problem with four circles.

Keywords: solution, circles, geometry

The problem under consideration is the following: *Given are three tangent to each other circles with radii a , b and $a+b$ respectively, where $a, b \in \mathbb{R}$ are positive real numbers. Find the radius of a fourth circle tangent to each of these three circles.*

Solution 1. Let the given circles be denoted as follows: $k_1(A, a)$, $k_2(C, b)$, $k_3(B, a+b)$ and $k_4(O, \rho)$; while $AC = a+b$, $BC = a+2b$, $AB = 2a+b$.



Let h, k, l be the altitudes of $\triangle ABC$ and let x, y, z be the distances from the point O to the sides BC, AC, AB of $\triangle ABC$.

We will use formula (1) from (Adamar, 1957, p. 564):

$$\frac{x}{2\rho} - \frac{h}{2(s-BC)} = l;$$

(where s is the semi-perimeter of $\triangle ABC$, i.e. $s = 2a + 2b$). Analogously,

$$\frac{y}{2\rho} - \frac{k}{2(s-AC)} = l \text{ and } \frac{z}{2\rho} - \frac{l}{2(s-AB)} = l, \text{ i.e.}$$

$$\frac{x}{2\rho} - \frac{h}{2a} = l; \frac{y}{2\rho} - \frac{k}{2(a+b)} = l; \frac{z}{2\rho} - \frac{l}{2b} = l,$$

$$\frac{x}{2\rho} - \frac{h}{2a} = l; \frac{y}{2\rho} - \frac{k}{2(a+b)} = l; \frac{z}{2\rho} - \frac{l}{2b} = l.$$

From here:

$$\frac{x}{h} - \frac{\rho}{a} = \frac{2\rho}{h}; \frac{y}{k} - \frac{\rho}{a+b} = \frac{2\rho}{k}; \frac{z}{l} - \frac{\rho}{b} = \frac{2\rho}{l},$$

$$\frac{x}{h} + \frac{y}{k} + \frac{z}{l} = \frac{\frac{x \cdot BC}{2}}{\frac{h \cdot BC}{2}} + \frac{\frac{y \cdot AC}{2}}{\frac{k \cdot AC}{2}} + \frac{\frac{z \cdot AB}{2}}{\frac{l \cdot AB}{2}} = \frac{F_{\triangle OBC} + F_{\triangle OAC} + F_{\triangle OAB}}{F_{\triangle ABC}} = \frac{F_{\triangle ABC}}{F_{\triangle ABC}} = 1:$$

$$(1) \quad l = \rho \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{b} + \frac{2}{h} + \frac{2}{k} + \frac{2}{l} \right)$$

We have:

$$F (= F_{\triangle ABC}) = \frac{(a+2b)h}{2} \Rightarrow \frac{2}{h} = \frac{a+2b}{F},$$

$$F (= F_{\triangle ABC}) = \frac{(a+b) \cdot k}{2} \Rightarrow \frac{2}{k} = \frac{a+b}{F},$$

$$F (= F_{\triangle ABC}) = \frac{(2a+b) \cdot l}{2} \Rightarrow \frac{2}{l} = \frac{2a+b}{F},$$

$$\text{where } F = \sqrt{(2a+2b) \cdot a \cdot b \cdot (a+b)} = (a+b)\sqrt{2ab}.$$

If follows now from (1), that:

$$\rho = \frac{I}{\frac{I}{a} + \frac{I}{a+b} + \frac{I}{b} + \frac{a+2b}{(a+b)\sqrt{2ab}} + \frac{a+b}{(a+b)\sqrt{2ab}} + \frac{2a+b}{(a+b)\sqrt{2ab}}},$$

$$\rho = \frac{I}{\frac{b(a+b)+ab+a(a+b)}{ab(a+b)} + \frac{a+2b+a+b+2a+b}{\sqrt{2ab}(a+b)}},$$

$$\rho = \frac{I}{\frac{a^2+b^2+3ab}{ab(a+b)} + \frac{2\sqrt{2ab}}{ab}} = \frac{I}{\frac{a^2+b^2+3ab+2\sqrt{2ab}(a+b)}{ab(a+b)}},$$

$$\rho = \frac{ab(a+b)}{a^2+b^2+3ab+2\sqrt{2ab}(a+b)}.$$

Remark 1. For the radius of the circle k' with center O' we get analogously:

$$\rho' = \left| \frac{F}{2s - (r_a + r_b + r_c)} \right|,$$

where $F (= F_{\triangle ABC}) = (a+b)\sqrt{2ab}$, $2s = 4(a+b)$, $r_a = \frac{F}{s - (a+2b)} = \frac{(a+b)\sqrt{2ab}}{a}$,

$$r_b = \frac{F}{s - (a+b)} = \frac{(a+b)\sqrt{2ab}}{a+b}; r_c = \frac{F}{s - (2a+b)} = \frac{(a+b)\sqrt{2ab}}{b}.$$

Solution 2. The semi-perimeter of $\triangle AOC$ is $\frac{I}{2}(a+b+a+\rho+b+\rho) = a+b+\rho$ and we have

$$\cos^2\left(\frac{I}{2}\angle AOC\right) = \frac{\rho(a+b+\rho)}{(a+\rho)(b+\rho)}; \quad \sin^2\left(\frac{I}{2}\angle AOC\right) = \frac{ab}{(a+\rho)(b+\rho)}.$$

We will use the following well known theorem (Grozdev, 2007):

If $\alpha' + \beta' + \gamma' = 180^\circ$, then

$$\sin^2 \alpha' - \sin^2 \beta' - \sin^2 \gamma' + 2 \sin \beta' \sin \gamma' \cos \alpha' = 0$$

and from here by $\alpha' = \frac{I}{2}\angle AOC$, $\beta' = \frac{I}{2}\angle BOC$ and $\gamma' = \frac{I}{2}\angle AOB$, we get:

$$\frac{ab}{(a+\rho)(b+\rho)} - \frac{b(a+b)}{(b+\rho)(a+b+\rho)} - \frac{a(a+b)}{(a+\rho)(a+b+\rho)} + 2 \cdot \frac{(a+b)\sqrt{ab\rho(a+b+\rho)}}{(a+b+\rho)(a+\rho)(b+\rho)} = 0, \text{ i.e.}$$

$$\frac{a+b+\rho}{a+b} - \frac{a+\rho}{a} - \frac{b+\rho}{b} + 2\sqrt{\frac{\rho(a+b+\rho)}{a+b}} = 0.$$

Further, we divide by ρ and use the substitutions $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{a+b}$ and $\delta = \frac{1}{\rho}$:

$$(*) \quad \alpha - \beta - \gamma - \delta + 2\sqrt{\beta\gamma + \gamma\delta + \delta\beta} = 0$$

Since

$$(\alpha + \beta + \gamma + \delta)^2 = (\alpha - \beta - \gamma - \delta)^2 + 4(\alpha\beta + \alpha\gamma + \alpha\delta)$$

$$\stackrel{(*)}{=} 4(\beta\gamma + \gamma\delta + \delta\beta) + 4(\alpha\beta + \alpha\gamma + \alpha\delta)$$

$$= 2(\alpha + \beta + \gamma + \delta)^2 - 2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$$

$$\Rightarrow 2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (\alpha + \beta + \gamma + \delta)^2,$$

We obtain:

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{(a+b)^2} + \frac{1}{\rho^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} + \frac{1}{\rho}\right)^2$$

$$\Leftrightarrow \frac{2}{\rho^2} + 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{(a+b)^2}\right) = \frac{1}{\rho^2} + 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b}\right) \cdot \frac{1}{\rho} + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b}\right)^2$$

$$\Leftrightarrow \frac{1}{\rho^2} - 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b}\right) \cdot \frac{1}{\rho} + 2\left(\frac{1}{(a+b)^2} + \frac{1}{a^2} + \frac{1}{b^2}\right) - \left(\frac{1}{a+b} + \frac{1}{a} + \frac{1}{b}\right)^2 = 0.$$

If $\frac{1}{\rho} = t$, then:

$$\begin{aligned}
 \Rightarrow t &= \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \pm \sqrt{2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \right)^2 - 2 \left(\frac{1}{(a+b)^2} + \frac{1}{a^2} + \frac{1}{b^2} \right)} \\
 \Rightarrow t &= \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \pm \sqrt{4 \left(\frac{1}{ab} + \frac{1}{a(a+b)} + \frac{1}{b(a+b)} \right)} \\
 \Rightarrow t &= \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \pm 2 \sqrt{\frac{a+b+b+a}{ab(a+b)}} \\
 \Rightarrow t &= \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \pm 2 \sqrt{\frac{2}{ab}}.
 \end{aligned}$$

We have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} = \frac{a+b}{ab} + \frac{1}{a+b} \stackrel{AM-GM}{\geq} 2 \sqrt{\frac{1}{ab}}.$$

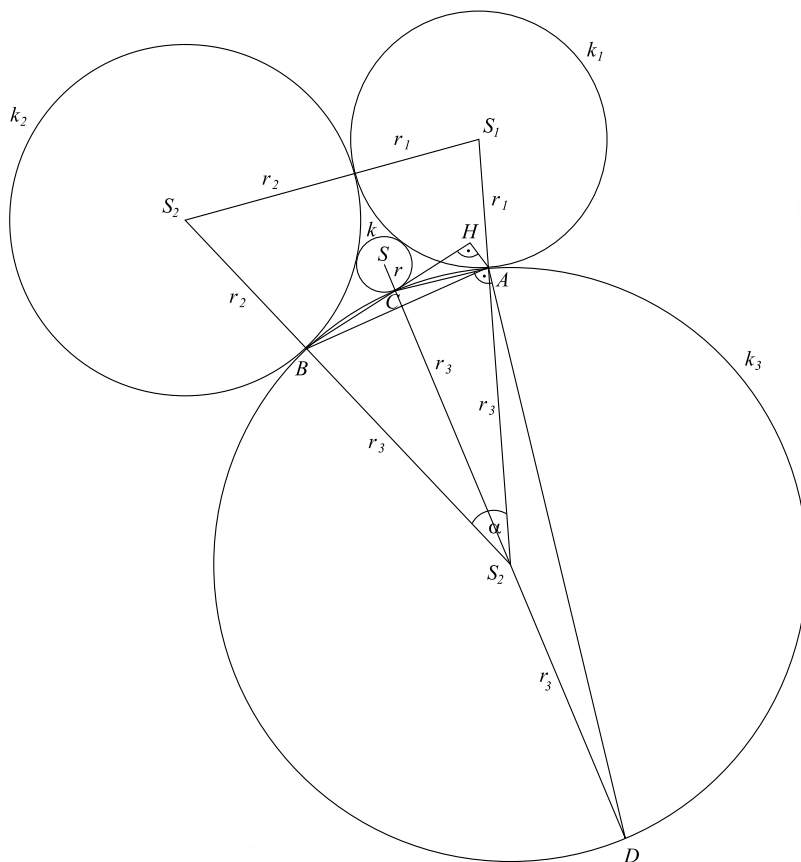
Since $2 \sqrt{\frac{1}{ab}} < 2 \sqrt{\frac{2}{ab}}$, then $t = \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} - 2 \sqrt{\frac{2}{ab}}$. It follows that there is only one solution:

$$\begin{aligned}
 t &= \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} + 2 \sqrt{\frac{2}{ab}}, \text{ i.e.} \\
 t &= \frac{b(a+b) + a(a+b) + ab}{ab(a+b)} + \frac{2\sqrt{2}}{\sqrt{ab}}, \\
 t &= \frac{a^2 + b^2 + 3ab}{ab(a+b)} + \frac{2\sqrt{2ab}}{ab}, \\
 t &= \frac{a^2 + b^2 + 3ab + 2\sqrt{2ab}(a+b)}{ab(a+b)}, \text{ i.e.} \\
 \rho &= \frac{ab(a+b)}{a^2 + b^2 + 3ab + 2\sqrt{2ab}(a+b)}.
 \end{aligned}$$

Solution 3. Let B and A be the tangent points of the circles k_2 and k_3 , k_1 and k_3 , respectively. By the law of cosines for $\triangle S_1AB$ we have:

$$(AB)^2 = (S_3B)^2 + (S_3A)^2 - 2 \cdot S_3B \cdot S_3A \cos \alpha, \text{ i.e.}$$

$$(2) \quad (AB)^2 = 2r_3^2 (1 - \cos \alpha).$$



Analogously, by the law of cosines for $\triangle S_1S_2S_3$:

$$\cos \alpha = \frac{(S_1S_3)^2 + (S_2S_3)^2 - (S_1S_2)^2}{2S_1S_3 \cdot S_2S_3}, \text{ i.e.}$$

$$\cos \alpha = \frac{(r_1 + r_3)^2 + (r_2 + r_3)^2 - (r_1 + r_2)^2}{2(r_1 + r_3)(r_2 + r_3)}$$

or

$$(3) \quad \cos \alpha = \frac{r_3^2 + r_1 r_3 + r_2 r_3 - r_1 r_2}{(r_1 + r_3)(r_2 + r_3)}.$$

Now, it follows from (2) and (3):

$$(4) \quad \begin{aligned} (AB)^2 &= 2r_3^2 \left(1 - \frac{r_3^2 + r_1 r_3 + r_2 r_3 - r_1 r_2}{(r_1 + r_3)(r_2 + r_3)} \right) \\ \Rightarrow (AB)^2 &= \frac{4r_1 r_2 r_3^2}{(r_1 + r_3)(r_2 + r_3)}. \end{aligned}$$

Let $k(S, r)$ be the fourth circle tangent to each of the circles k_1, k_2 and k_3 . Let A, B and C be the tangent points of circles k_1 and k_3, k_2 and k_3, k_3 and k , respectively. Let the point H be the foot of the normal from the point A to the line BC . Using (4) for the circles k_1, k_2, k_3 firstly, then for the circles k_2, k_3, k and finally for the circles k_1, k_3, k , we get:

$$(5) \quad \begin{aligned} (AB)^2 &= \frac{4r_1 r_2 r_3^2}{(r_1 + r_3)(r_2 + r_3)}, \\ (BC)^2 &= \frac{4r_2 r_3^2 r}{(r_2 + r_3)(r + r_3)}, \end{aligned}$$

$$(6) \quad (AC)^2 = \frac{4r_1 r_3^2 r}{(r_1 + r_3)(r + r_3)}.$$

The similarity of the right triangles $\triangle ABH$ and $\triangle CDA$ gives:

$$\frac{AC}{CD} = \frac{AH}{AB},$$

and from here by (4) and (6), we obtain:

$$(7) \quad (AH)^2 = \frac{(AB)^2 \cdot (AC)^2}{4r_3^2} = \frac{4r_1^2 r_2 r_3^2 r}{(r_1 + r_3)^2 (r_2 + r_3)(r + r_3)},$$

$$(8) \quad (CH)^2 = (AC)^2 - (AH)^2 = \frac{4r_1 r_3^2 r}{(r_1 + r_3)(r + r_3)} \left(1 - \frac{r_1 r_2}{(r_1 + r_3)(r_2 + r_3)} \right).$$

By (4), (5), (6), (7) and (8):

$$\begin{aligned}(AB)^2 &= (BH)^2 + (AH)^2 = (BC + CH)^2 + (AH)^2 = (BC)^2 + 2BC \cdot CH + (CH)^2 + (AH)^2 = \\ &= (AC)^2 + (BC)^2 + 2BC \cdot CH, \text{ i.e.}\end{aligned}$$

$$\begin{aligned}\frac{4r_1r_2r_3^2}{(r_1+r_3)(r_2+r_3)} &= \frac{4r_1r_3^2r}{(r_1+r_3)(r+r_3)} + \frac{4r_2r_3^2r}{(r_2+r_3)(r+r_3)} + 2 \cdot \frac{2r_3\sqrt{r_2r}}{\sqrt{(r_2+r_3)(r+r_3)}} \cdot \\ &\cdot \frac{2r_3\sqrt{r_1r}}{\sqrt{(r_1+r_3)(r+r_3)}} \sqrt{1 - \frac{r_1r_2}{(r_1+r_3)(r_2+r_3)}}.\end{aligned}$$

We get: $r_1r_2(r+r_3) = r_1r(r_2+r_3) + r_2r(r_1+r_3) + 2r\sqrt{r_1r_2r_3(r_1+r_2+r_3)}$

and finally $r = \frac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_1r_3 + 2\sqrt{r_1r_2r_3(r_1+r_2+r_3)}}.$

If $r_1 = a, r_2 = b, r_3 = a + b$, then:

$$r = \frac{ab(a+b)}{ab+b(a+b)+a(a+b)+2\sqrt{ab(a+b)(2a+2b)}}, \text{ i.e.}$$

$$r = \frac{ab(a+b)}{a^2+b^2+3ab+2(a+b)\sqrt{2ab}}.$$

REFERENCES

1. Adamar, Ž. (1957). *Elementarnaja geometrija, Knjiga 1., Planimetrija*. Moskva.
2. Grozdev, S. (2007). *For High Achievements in Mathematics. The Bulgarian*
3. *Experience (Theory and Practice)*. Sofia: ADE. (ISBN 978-954-92139-1-1), 295 pages.

✉ **Prof. Dr. Šefket Arslanagić**

Faculty of Mathematics and Natural Sciences

University of Sarajevo

Sarajevo, Bosna and Herzegovina

E-mail: asefket@pmf.unsa.ba