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Научно-методически статии

THE ROLE OF COMPUTER ANIMATION IN MATHEMATICS TEACHING

Sergey Larin, Valeriy Mayer
Krasnoyarsk State Pedagogical University (Russia)

Abstract. The article is devoted to the role, importance, and use of animated drawings created on computer screens in GeoGebra platform for Mathematics teaching at schools and Universities. The use of this platform introduces a new kind of visibility in Mathematics learning, which entails not only an increased interest to the subject, but also promotes understanding and stronger perception of abstract material. In addition, an opportunity appears to remove computing difficulties, since the calculations are done by the computer. The user becomes a manager of computing and this saves time, but also allows acquiring new competence in useful digital technologies in education. Animated graphics help to realize experiments, to notice new patterns, to suggest solutions of mathematical problems and to illustrate them successfully. All that is of great importance for the formation of the new generation, called upon the development of the Russian digital economy.

Keywords: entertainment figure; GeoGebra; computational problem; limaçon (Pascal's snail); quadratic polynomial in the complex plane

Introduction. Computers emerged as instruments to accelerate calculations with numbers. Therefore, primary the use of computers in education should be expected to be calculation assistance. Calculators are already firmly established in computing practice. However, a simple calculator can not even process an ordinary fraction, replacing it by its decimal approximation. Modern computers manipulate not only with numbers, but also with formulae, using symbolic computation. A full mastering of computer technologies in the educational practice is an urgent task for modern educators. We want to create educational products that aim at acquisition of certain mathematical knowledge rather than simple calculations, putting this task to computers. Relevant examples are proposed in the sequel.

Another task, which expects assistance of computers, is to make teaching of Mathematics more visual. The problem of visibility in Mathematics is as old as the world is. The problem is to make abstract mathematical concepts and statements more concrete. A large number of methodologists address to it. Using computer animation is a new solution that has appeared fairly recently, thanks to the development of computer technologies.

Computer animation, providing visuals, brings movements into Mathematics teaching (Larin, 2015). It provides ability to simulate specified movements by functions and to graph them simultaneously. Movements combine Physics, which studies different kinds of movements, Mathematics as means of studying movements, and also Computer science as means to model movements.

A third direction of the computer use in education is connected with the possibility to realize experiments (Sergeeva, Shabanova & Grozdev, 2014), finding solutions of mathematical problems. Before, in order to formulate and prove a theorem it was necessary to assume it being true on experimental basis. Now, computer experimentations become essential part of the technology in mathematical problem solving. Animated features of dynamical systems of Mathematics, such as GeoGebra[2)], could be the basis of experimentation.

We distinguish the following three kinds of computer animation:

1) Geometric animations based on maintaining consistency in building animated pictures when one of its independent (primary) elements is put in motion: points, straight lines, circles and so on.

2) Algebraic animations – controlled changes in formula settings.

3) Text animations – controlled text modifications under visibility conditions.

In practice, these three types of animation are used together, complementing and enriching one another.

Algebraic animations. The use of animated figures is effective in addressing the following methodological issues. Secondary and post-secondary Mathematics curricula contain many computational algorithms that students must understand. During demonstration of computational algorithms and their optimization there often arise cumbersome computing difficulties, distracting students from the main objective of mastering the algorithm itself. Such a methodological problem could be solved in two ways. One could either charge the computer with the calculations, or create a task with a predictable solution without computing difficulties. Both solutions are implemented using animated figures.

As an example of correct computing difficulties refer to animated figure 1 performing the well-known Euclidean division.

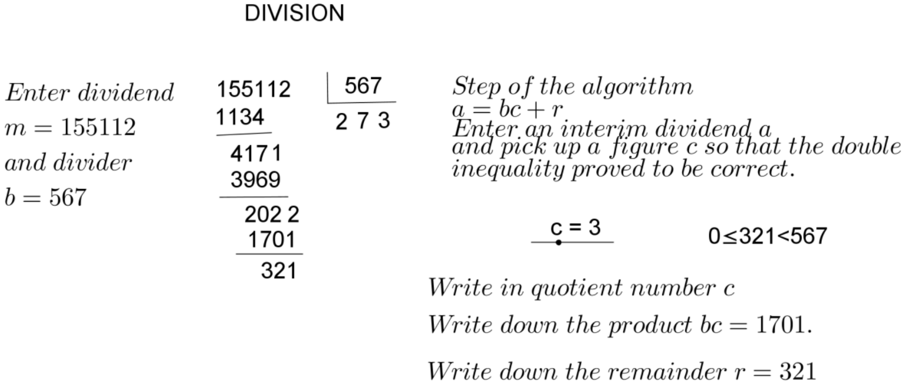

Figure 1

Here the selection of each quotient of a number (under the corner) is carried out by moving the point on the slider. We seek to make the double inequality to be true (changing the *c* changes the middle part of the inequality, which is the modul, which must be within the specified bounds). When selecting the quotient, the computer generates numbers that are the product of the number and the numerator as well as the modulo, so all you have to do is use the ABC button and place into the correct space: the number under the corner, the product of the number and the numerator above the line, and the modulo below the line.

FINDING a PAIR of INTEGERS
with given g.c.d. and predetermined sequence
of fractional private

$Enter\ d = g.c.d(a, b),$
$the\ planned\ number\ of\ steps$
$n\ Evclid's\ algoritm\ and$
$the\ sequence\ incomplete\ q_1,\ ...\ q_n.$

n=4 $d = 53,\ \ q_1 = 78,\ \ q_2 = 87\ \ q_3 = 23,\ \ q_4 = 5$

$Enter:$
$r_3 = d,$
$r_2 = d \cdot q_4,$
$r_1 = r_2 \cdot q_3 + r_3,$
$b = r_1 \cdot q_2 + r_2,$
$a = b \cdot q_1 + r_1.$

$Answer:$
$a = 41747146,$
$b = 535141$

Figure 2

As you know, the division above is a step the Euclidean algorithm for finding the GCD. This allows to create an animated figure in GeoGebra platform which implements the Euclidean algorithm while eliminating the necessity of calculations, which are now performed by the computer.

Animated figure 2 allows you to get a pair of natural numbers with given GCD and an invented sequence of incomplete quotients during the implementation of Euclidean algorithm. With such pairs of numbers we can be safely demonstrate the Euclidean algorithm, with no surprises with calculations.

Let’s note that the animated figures can be easily converted to use with polynomials.

In accordance with the curriculum of “algebra and theory of numbers” at a pedagogical university, students have to solve problems finding rational roots of polynomials with integer coefficients. This task is also addressed in the secondary school program in an advanced mathematics class. The theory of this question is quite simple. However, the implementation sometimes rests in large computational difficulties, so it naturally begs the aid to the computer.

Animated Figure 3 can take its rightful place in the lessons of mathematics. Consider the use of this figure.

RATIONAL ROOTS

$$f(x) = 15\,x^4 + 29\,x^3 + 67\,x^2 - 17\,x - 30$$

Ehter the module free Member m

Enter senior module quotient n

Candidates in the numerator is the number of root from the list $\{1, 2, 3, 5, 6, 10, 15, 30\}$ *and among them the opposite*

Candidates in the denominator of the root $\{\mathrm{n}, 3, 5, 15\}$

Enter a candidate a numerator root and candidate b in the denominator of the root.

$$Number\ \frac{a}{b} = \frac{-3}{5}\ is\ the\ root.$$

Figure 3

1) Enter the polynomial $f(x) = 15x^4 + 29x^3 + 67x^2 - 17z - 30$.

2) Enter the module of the free term $m = 30$. A list of all divisors of the number m appears on the canvas

3) Enter the module of the quadratic coefficient $n = 15$. A list of all divisors of the number n appears on the canvas.

4) Enter the next contender to the denominator of the root (from the second list), starting with $b = 1$, and for selected denominator, alternately enter the root

numerator candidates (numbers from the first list and opposite to it), starting with $a=1$, $a=-1$, $a=2$, $a=-2$,... The characteristic of the fraction $\frac{a}{b}$, whether is it the root or not, appears on the canvas.

To reduce the number of tests, use the following fact. If the irreducible fraction with integer coefficients $\frac{a}{b}$ is the root of this polynomial $f(x)$, then for any integer *k*, the number $\frac{f(k)}{bk-a}$ must be an integer. In particular, the number $\frac{f(1)}{b-a}$ and $\frac{f(-1)}{b+a}$ must be integers. Proving this fact is complicated. However, the computer easily finds the value $f\left(\frac{a}{b}\right)$, so the check is not necessary. Thus, new technologies in teaching led to a revision of which basic knowledge is essential.

Within the framework of dynamic mathematics many concepts and theorems become "visible" and "tangible" to the students. Along the way, the student learns to use computer technology not only in education but also in solving research problems.

Solving mathematical tasks in a dynamic geometric platform passes through three stages:

1) Geometric modeling of the conditions of tasks on your computer screen.
2) Solving the problem on the screen using animations.
3) Building the mathematical model of the solution seen on the screen.

Geometric animations. Let's look at one observation that led to a mathematical assertion as an example of using geometric animation in the study. Students from Arkhangelsk, within the framework of the project "We write (Pisemsami)"[1)] in accordance with the tasks set out by V. R Meyer in the "matrix article", researched the curve called the limaçon of Pascal. In the book by S.V. Larin (Larin, 2015), they found pictures of images of the unit circle within the projections specified by the second degree polynomials in the complex plane, and noticed their resemblance with the limaçon of Pascal. They proved that the second degree polynomial $z^2+k(1+i)z$ of a complex variable *z*, whose module is equal to 1, describes the limaçon of Pascal, and suggested that this observation will always be true. Below we give a mathematical justification of these observations.

Let's consider the kinematic definition of the limaçon. Let's suppose that a circle rolls on the outside of another circle of the same radius (fig. 4). Let's fix a certain point on the outer circle. The point, rolling on the outer circle, leaves a path, which is called the cardioid (fig. 4*a*). Now, let's draw a ray through the center of the circle and the rolling point that traces the cardioid; let's also mark a point D. While rolling on the outer circle, the point D will trace a path, which is called the limaçon of Pascal (fig. 4*b*, 4*c*). If the radii of the circles are equal r, and the distance from the center of the circle to the point D is equal R then the number $d = R - r$ is called the deviation of the limaçon of Pascal from the cardioid. If $d > 0$ ($d < 0$), then the limaçon of Pascal will be called the elongated (shortened) cardioid. The Figure 4*b* shows the elongated and Figure 4*c* the shortened cardioid. When $d = 0$, the limaçon of Pascal turns into the cardioid.

Note that the limaçon of Pascal turns in a circle if and only if $R = 0$ and $d = -r$. This is the minimum possible value. Also note that the limaçon of Pascal is uniquely identified by specifying the fixed circle and deviations d.

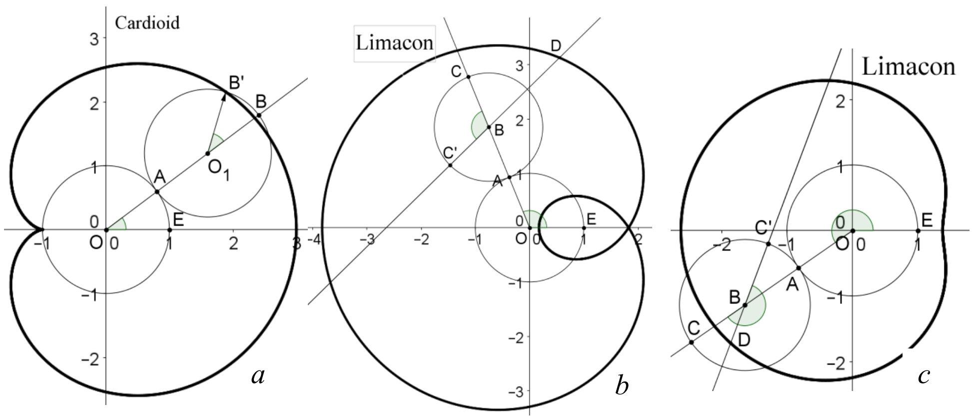

Figure 4

Now let's build a unit circle on the complex plane, mark point Z on it, representing a complex number, mark complex numbers $a \neq 0, b, c$ with other points, and write the expression $w = aZ^2 + bZ + c$ on the input line. After entering, a point depicting the complex number w, appears on the canvas. Let's make the point w leave a path and enable the animation of the point Z. As a result, the point w will trace the image of the unit circle when displaying the specified data from the second degree polynomial (fig. 5*a*). This graph resembles the limaçon of Pascal. Is this observation true?

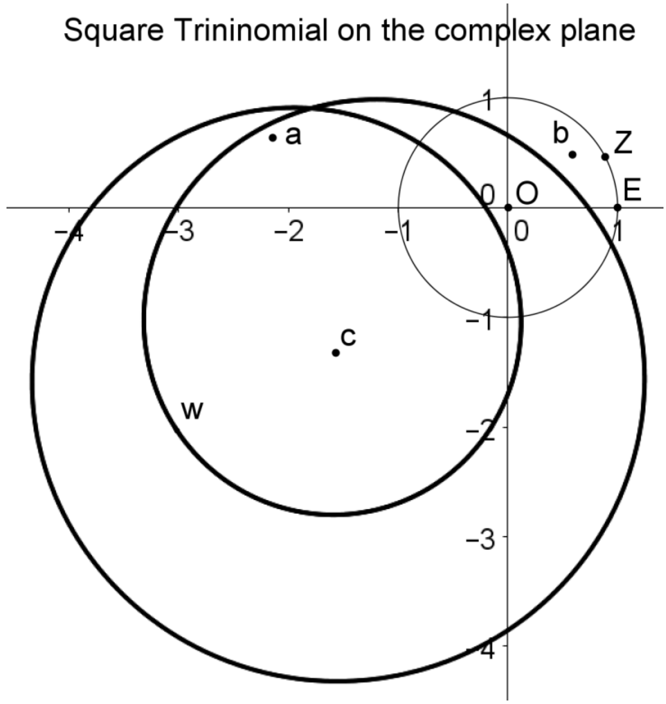

Figure 5

During the experiment, using the movement of the points that depict the polynomials, we change the polynomial coefficients. In any case, it looks like we get the confirmation of our guesses. Let's prove that our observations are true.

Theorem 1. *On the complex plane, the limaçon of Pascal that has the unit circle as the fixed point and deviation d from the cardioid is the image of the unit circle, with the display of the complex plane, specified by a polynomial* $w=(d+1)z \ \ +2z$.

Proof. Let O and O_1 respectively be centres of the fixed and rolling circles, and let point D plot the limaçon (fig. 4*b*, 6). Let's introduce the rectangular coordinate system, with the center of the fixed circle O as the origin. The x-axis has the starting position on the line OO_1 where the points O, O_1 and D are collinear. The radius of the circle is going to be one unit on the axes. Let's perform the following additions in Figure 4*b* (see Figure 6).

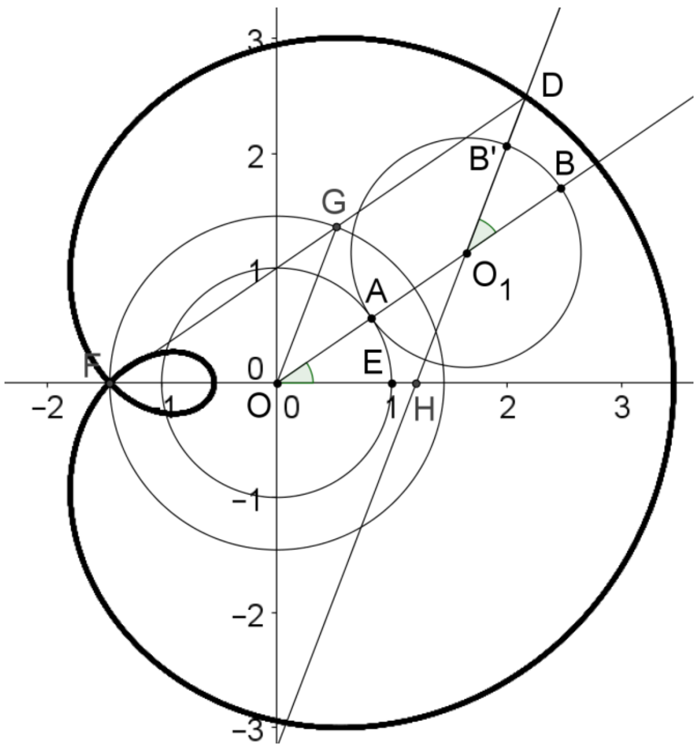

Figure 6

According to the initial statement, $B'D = d$. Let's construct a point $F = (-d-1, 0)$, then draw a circle with the center at the origin, and passing through the point F. Draw a line segment DF and mark the point G which is the intersection point of the line segment and the circle; perform the rest of the building captured in Figure 6. Let the marked angles be equal φ and $\angle GFO = \psi$. Let's prove that $\psi = \varphi$ and the quadrangle $BDGO$ is parallelogram (students working within the framework of the above-mentioned project proved this, but for completeness and independence we'll cite our own proof). Looking at the vertical angles, we realise that $\angle OO_1C = \varphi$. Consequently, the triangle ΔOO_1C is an isosceles triangle and $OC = O_1C$. But then $FC = DC$ and the triangle ΔDFC is an isosceles triangle, where $\angle FDC = \angle DFC = \psi$, and since the triangle ΔFGO is an isosceles triangle ($FO = OG$), then $\angle FGO = \psi$. Hence, $\angle FGO = \angle FDC$ and $GO \parallel DO_1$. But $GO = DO_1$, therefore the quadrangle $BDGO$ is a parallelogram. But then the lines FD and OB are parallel and $\psi = \angle GDO_1 = \angle DO_1B = \varphi$.

Next, $\overrightarrow{OD} = \overrightarrow{OG} + \overrightarrow{OB}$. Let points G and D depict complex numbers z and w respectively. Then $\overrightarrow{OG} = (d+1)z^2$, $\overrightarrow{OB} = 2z$ and the vector equality becomes $w = (d+1)z^2 + 2z$. Therefore, the limaçon is a unit circle when displaying the specified derived polynomial. The theorem is proved.

Note that if $d = -1$, then the limaçon turns into a circle, and the resulting polynomial into a first degree a polynomial $w = 2z$.

Theorem 2. *On the complex plane each second degree polynomial displays a projection, where the image of the unit circle is a limaçon, with a suitable choice of the fixed circle and deviations from the cardioid.*

Proof. Let's consider the display of a complex plane that is set by a square trinomial $w = az^2 + bz + c$ with complex coefficients. Assuming that $b = 0$, we get a polynomial $w = az^2 + c$. In the animated Figure 5 point b is placed at the origin O. Now we see that the thick circle with the center in point c, that represents the free member of a polynomial, is the image of the unit circle set by the aforementioned polynomial (fig. 7).

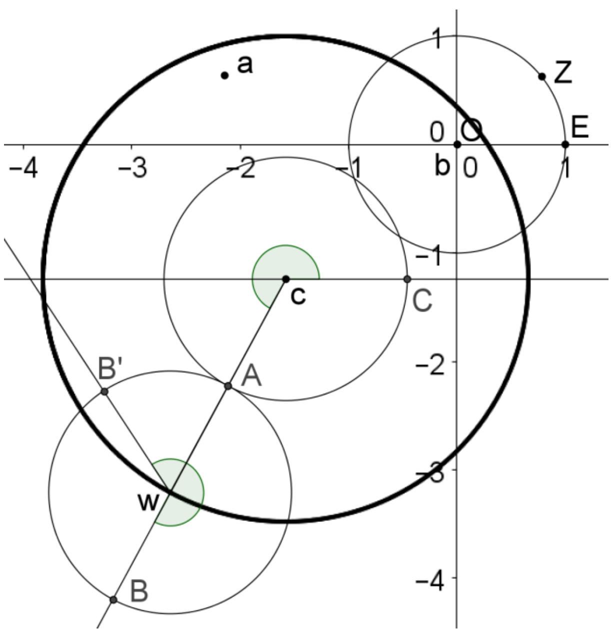

Figure 7

Let's draw a ray cw with the point *A* being the middle of the segment cw, then draw two circles with points c and w being the respective centers and the radius being $r = cA$. Let's draw a horizontal line through point c and construct an angle $\angle BwB' = \angle CcA$. Let's draw a ray wB' that will help track the rolling of the circle with the center on the point w on the fixed circle with its centre at the point c. The point w traces the limaçon with the deviation $d = -r$ which coincides with the thickened circle.

Consider the case where $b \neq 0$. Using parallel migration without changing the appearance of the image of the unit circle, we can get rid of the free member and consider this square trinomial in the form $w = aZ^2 + bZ$. Let's consider the trigonometric forms of the coefficient and the variable: $a = k(\cos\alpha + i\sin\alpha)$, $b = m(\cos\beta + i\sin\beta)$, $Z = \cos\varphi + i\sin\varphi$.

Then we get

$$w = k(\cos\alpha + i\sin\alpha)(\cos\varphi + i\sin\varphi)^2 + m(\cos\beta + i\sin\beta)(\cos\varphi + i\sin\varphi) =$$
$$= k(\cos(\alpha + 2\varphi) + i\sin(\alpha + 2\varphi)) + m(\cos(\beta + \varphi) + i\sin(\beta + \varphi)).$$

Let's replace the variable considering that $\varphi = \psi - \alpha/2$. We get $w_1 = k(\cos 2\psi + i\sin 2\psi) + m(\cos\psi + i\sin\psi)(\cos(\beta - \alpha/2) + i\sin(\beta - \alpha/2)$.

Let's multiply the last equality by $\cos(\alpha - 2\beta) + i\sin(\alpha - 2\beta)$. This will lead the the curve made by point w_1 to turn if we change ψ, but that does not change the appearance of the curve. At the same time we get a polynomial

$$w_2 = w_1(\cos(\alpha - 2\beta) + i\sin(\alpha - 2\beta)) =$$
$$= k(\cos(2\psi + \alpha - 2\beta) + i\sin(2\psi + \alpha - 2\beta)) +$$
$$+ m(\cos(\psi + \alpha/2 - \beta) + i\sin(\psi + \alpha/2\alpha - \beta)) = kz_1^2 + mz_1,$$

where $z_1 = \cos(\psi + \alpha/2 - \beta) + i\sin(\psi + \alpha/2\alpha - \beta)$. Assuming $z_1 = \dfrac{2}{m} z_2$, $d = \dfrac{4k}{m^2} - 1$, we get a polynomial $w_2 = (d+1)z_2^2 + 2z_2$. According to Theorem 1 the limaçon with a fixed unit circle and deviation d is the image of the unit circle when displaying the specified latest polynomial. The theorem is proved.

Conclusion. This way digital technology in education is included naturally into the arsenal of teaching tools of a school teacher and a university professor. Already the examples above suggest that digital technologies in education are shaping the new face of modern didactics.

NOTES

1. Pisemsami: official site [electronic resource]. Access mode: https://sites.google.com/site/pisemsami/home
2. GeoGebra: official site [electronic resource]. Access mode: http://www.geogebra.org

REFERENCES

Larin, S. V. (2015). *Computer animation in GeoGebra platform in mathematics classes*. Rostov-na-Donu: Legion.

Sergeeva, T., M. Shabanova & S. Grozdev (2014). *Foundations of Dynamic Geometry*. Moscow: ASOU (in Russian). [Сергеева, Т., М. Шабанова & С. Гроздев (2014). *Основы динамической геометрии*. Москва: АСОУ.]

✉ **Prof. Sergey Larin, DSc.**
✉ **Prof. Valeriy Mayer, DSc.**
School of Mathematics, Physics and Computer Technology
Krasnoyarsk State Pedagogical University
Krasnoyarsk, Russia
E-mail: larin_serg@mail.ru
E-mail: mavr49@mail.ru