Educational Matters
Научно-методически статии

THE REARRANGEMENT INEQUALITY

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Abstract. In this paper we consider a really very useful inequality, the so called rearrangement inequality, which has may applications and could be used in proving other inequalities. The paper contains a proof of the rearrangement inequality and several examples of its application.

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Theorem 1. Consider two collections of real numbers in increasing order:

$$a_1 \le a_2 \le \ldots \le a_n \text{ and } b_1 \le b_2 \le \ldots \le b_n .$$

For any permutation $\left(a'_1, a'_2, \ldots, a'_n\right)$ of $\left(a_1, a_2, \ldots, a_n\right)$, it happens that

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \ge a'_1 b_1 + a'_2 b_2 + \ldots + a'_n b_n \quad (1)$$

$$\ge a_n b_1 + a_{n-1} b_2 + \ldots + a_1 b_n . \quad (2)$$

Moreover, the equality in (1) holds true iff $\left(a'_1, a'_2, \ldots, a'_n\right) = \left(a_1, a_2, \ldots, a_n\right)$ and the equality in (2) holds true iff $\left(a'_1, a'_2, \ldots, a'_n\right) = \left(a_n, a_{n-1}, \ldots, a_1\right)$. (1) is known to be the **rearrangement inequality**.

Proof of the rearrangement inequality: Suppose that $b_1 \le b_2 \le \ldots \le b_n$. Let

$$S = a_1 b_1 + a_2 b_2 + \ldots + a_r b_r + \ldots + a_s b_s + \ldots + a_n b_n ,$$

$$S' = a_1 b_1 + a_2 b_2 + \ldots + a_s b_r + \ldots + a_r b_s + \ldots + a_n b_n .$$

The difference between S and S' is that the coefficients of b_r and b_s, where $r < s$, are switched. Hence,

$$S-S'=a_rb_r+a_sb_s-a_sb_r-a_rb_s=(b_s-b_r)(a_s-a_r).$$

Thus, we have that $S\ge S'$ iff $a_s\ge a_r$. Repeating this process we get as a result that the sum S is maximal when $a_1\le a_2\le\ldots\le a_n$.

Corollary 1. For any permutation $(a_1',a_2',\ldots,a_n')$ of $(a_1,a_2,\ldots,a_n)$, we have that

$$a_1^2+a_2^2+\ldots+a_n^2\ge a_1a_1'+a_2a_2'+\ldots+a_na_n'.$$

Corollary 2. For any permutation $(a_1',a_2',\ldots,a_n')$ of $(a_1,a_2,\ldots,a_n)$, we have that

$$\frac{a_1'}{a_1}+\frac{a_2'}{a_2}+\ldots+\frac{a_n'}{a_n}\ge n.$$

In the sequel we propose several examples of application of the rearrangement inequality.

Example 1. Let a,b and c be positive real numbers. Prove **Nesbitt's inequality**

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}.$$

Solution: Without loss of generality, we may assume that $a\ge b\ge c$. Then clearly

$$\frac{1}{b+c}\ge\frac{1}{c+a}\ge\frac{1}{a+b}.$$

By the rearrangement inequality we deduce

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}$$

and

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}.$$

Adding the last two inequalities we obtain that

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\ge 3$$

or

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}\text{, q.e.d.}$$

Equality holds true iff $a=b=c$.

Example 2. (IMO 1978.) Let $x_1,x_2,\ldots,x_n$ be different positive integers. Prove the inequality

$$\frac{x_1}{1^2}+\frac{x_2}{2^2}+...+\frac{x_n}{n^2}\geq\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}.$$

Solution: Let $(a_1,a_2,...,a_n)$ be a permutation of $(x_1,x_2,...,x_n)$ with $a_1\leq a_2\leq...\leq a_n$ and $(b_1,b_2,...,b_n)=\left(\frac{1}{n^2},\frac{1}{(n-1)^2},...,\frac{1}{1^2}\right)$, that is $b_i=\frac{1}{(n+1-i)^2}$ for $i=1,2,...,n$.

Consider the permutation $(a_1',a_2',...,a_n')$ of $(a_1,a_2,...,a_n)$ defined by $a_i'=x_{n+1-i}$, for $i=1,2,...,n$. Using inequality (2) we may claim that

$$\begin{aligned}\frac{x_1}{1^2}+\frac{x_2}{2^2}+...+\frac{x_n}{n^2}&=a_1'b_1+a_2'b_2+...+a_n'b_n\\ &\geq a_nb_1+a_{n-1}b_2+...+a_1b_n\\ &=a_1b_n+a_2b_{n-1}+...+a_nb_1\\ &=\frac{a_1}{1^2}+\frac{a_2}{2^2}+...+\frac{a_n}{n^2}.\end{aligned}$$

Since $1\leq a_1,\ 2\leq a_2,...,n\leq a_n$ we have that

$$\frac{x_1}{1^2}+\frac{x_2}{2^2}+...+\frac{x_n}{n^2}\geq\frac{a_1}{1^2}+\frac{a_2}{2^2}+...+\frac{a_n}{n^2}\geq\frac{1}{1^2}+\frac{2}{2^2}+...+\frac{n}{n^2}=\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n},\text{ q.e.d.}$$

Equality holds true iff $x_1=1,\ x_2=2,...,x_n=n$.

Example 3. Let a,b,c be positive real numbers. Prove the inequality

$$\frac{a^2+c^2}{b}+\frac{b^2+a^2}{c}+\frac{c^2+b^2}{a}\geq 2(a+b+c).$$

Solution: Since the given inequality is symmetric, without loss of generality we may assume that $a\geq b\geq c$. Then clearly

$$a^2\geq b^2\geq c^2 \text{ and } \frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a}.$$

By the rearrangement inequality we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=a^2\cdot\frac{1}{b}+b^2\cdot\frac{1}{c}+c^2\cdot\frac{1}{a}\geq a^2\cdot\frac{1}{a}+b^2\cdot\frac{1}{b}+c^2\cdot\frac{1}{c}=a+b+c, \tag{3}$$

and

$$\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}=a^2\cdot\frac{1}{c}+b^2\cdot\frac{1}{a}+c^2\cdot\frac{1}{b}\geq a^2\cdot\frac{1}{a}+b^2\cdot\frac{1}{b}+c^2\cdot\frac{1}{c}=a+b+c. \tag{4}$$

Adding (3) and (4) yields the required inequality. The equality occurs iff $a=b=c$.

Example 4. Let x,y,z be positive real numbers. Prove the inequality

$$\frac{x^3}{yz}+\frac{y^3}{zx}+\frac{z^3}{xy}\geq x+y+z\ .$$

Solution: Since the given inequality is symmetric we may assume that $x\geq y\geq z$. Then

$$x^3\geq y^3\geq z^3 \text{ and } \frac{1}{yz}\geq\frac{1}{zx}\geq\frac{1}{xy}\ .$$

By the rearrangment inequality we have

$$\frac{x^3}{yz}+\frac{y^3}{zx}+\frac{z^3}{xy}=x^3\cdot\frac{1}{yz}+y^3\cdot\frac{1}{zx}+z^3\cdot\frac{1}{xy}\geq x^3\cdot\frac{1}{xy}+y^3\cdot\frac{1}{yz}+z^3\cdot\frac{1}{zx}=\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\ . \quad (5)$$

We will prove that

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z\ . \quad (6)$$

Let $x\geq y\geq z$. Then $x^2\geq y^2\geq z^2$ and $\frac{1}{z}\geq\frac{1}{y}\geq\frac{1}{x}$ (since inequality (6) is cyclic, we need to consider the case $z\geq y\geq x$).

By the rearrangment inequality we obtain

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq\frac{x^2}{x}+\frac{y^2}{y}+\frac{z^2}{z}=x+y+z\ .$$

The case when $z\geq y\geq z$ is analogous to the previous one.

Now by (5) and (6) we obtain

$$\frac{x^3}{yz}+\frac{y^3}{zx}+\frac{z^3}{xy}\geq x+y+z\ \text{, q.e.d.}$$

Equality occurs iff $x=y=z$.

Example 5. (IMO 1964). Suppose that a,b,c are the lenghts of the sides of a triangle. Prove that

$$a^2\left(b+c-a\right)+b^2\left(a+c-b\right)+c^2\left(a+b-c\right)\leq 3abc\ .$$

Solution: Since the expression is a symmetric function of a,b and c, we can assume, without loss of generality, that $c\leq b\leq a$. In this case, $a\left(b+c-a\right)\leq b\left(a+c-b\right)\leq c\left(a+b-c\right)$. For instance, the first inequality could be proved in the following way:

$$a\left(b+c-a\right)\leq b\left(a+c-b\right)\Leftrightarrow ab+ac-a^2\leq ab+bc-b^2$$

$$\Leftrightarrow (a-b)c \le (a+b)(a-b)$$

$$\Leftrightarrow (a-b)(a+b-c) \ge 0 .$$

By (2) of the rearrangement inequality, we have

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \le bc(b+c-a)+cb(c+a-b)+ac(a+b-c),$$

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \le ca(b+c-a)+ab(c+a-b)+bc(a+b-c).$$

Therefore, if follows after summing these inequalities, that:

$$2\left[a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\right] \le 6abc ,$$

i.e.

$$a^2(b+c-a)+b^2(a+c-b)+c^2(a+b-c) \le 3abc \text{, q.e.d.}$$

Equality holds iff $a=b=c$ (equilateral triangle).

Example 6. (IMO 1983). Let a,b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \ge 0 .$$

Solution: Consider the case $c \le b \le a$ (the other casses are similar). As in the previous example, we have that

$$a(b+c-a) \le b(a+c-b) \le c(a+b-c)$$

and since $\frac{1}{a} \le \frac{1}{b} \le \frac{1}{c}$, using the inequality (1), we deduce, that:

$$\frac{1}{a}\cdot a(b+c-a)+\frac{1}{b}\cdot b(c+a-b)+\frac{1}{c}\cdot c(a+b-c) \ge \frac{1}{c}\cdot a(b+c-a)+\frac{1}{a}\cdot b(c+a-b)+\frac{1}{b}\cdot c(a+b-c).$$

Therefore,

$$a+b+c \ge \frac{a(b-a)}{c}+\frac{b(c-b)}{a}+\frac{c(a-c)}{b}+a+b+c .$$

It follows that

$$\frac{a(b-a)}{c}+\frac{b(c-b)}{a}+\frac{c(a-c)}{b} \le 0 .$$

Multiplying by abc, we obtain

$$a^2b(b-a)+b^2c(b-c)+c^2a(c-a) \ge 0 \text{, q.e.d.}$$

Remark 1. Four different proofs of this inequality could be found in (Arslanagić, 2005).

Example 7. Let $a_1, a_2, ..., a_n$ be distinct positive integers. Show that

$$\frac{a_1}{2} + \frac{a_2}{8} + ... + \frac{a_n}{n \cdot 2^n} \geq 1 - \frac{1}{2^n} .$$

Solution: Arrange $a_1, a_2, ..., a_n$ in increasing order as $b_1, b_2, ..., b_n$. Then $b_m \geq n$ because we have distinct positive integers. Since $\frac{1}{2}, \frac{1}{8}, ..., \frac{1}{n \cdot 2^n}$, by the rearrangement inequality it follows that:

$$\frac{a_1}{2} + \frac{a_2}{8} + ... + \frac{a_n}{n \cdot 2^n} \geq \frac{b_1}{2} + \frac{b_2}{8} + ... + \frac{b_n}{n \cdot 2^n}$$

$$\geq \frac{1}{2} + \frac{2}{8} + ... + \frac{n}{n \cdot 2^n}$$

$$= \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{2^{n-1}}\right)$$

$$= \frac{1}{2} \cdot \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n} \text{, q.e.d.}$$

Equality holds true iff $a_1 = 1, a_2 = 2, ..., a_n = n$.

Example 8. (West German Math Olympiad, 1982). If $a_1, a_2, ..., a_n > 0$ and $a = a_1 + a_2 + ... + a_n$; then

$$\sum_{i=1}^{n} \frac{a_i}{2a - a_i} \geq \frac{n}{2n - 1} .$$

Solution: By symmetry we may assume that $a_1 \geq a_2 \geq ... \geq a_n$. Then

$$\frac{1}{2a - a_n} \leq ... \leq \frac{1}{2a - a_1} .$$

For convenience let $a_i = a_j$ if $i \equiv j \left(mod\, n\right)$. For $m = 0, 1, ..., n-1$ by the rearrangement inequality we get

$$\sum_{i=1}^{n} \frac{a_{m+i}}{2a - a_i} \leq \sum_{i=1}^{n} \frac{a_i}{2a - a_i} .$$

Adding these n inequalities we have

$$\sum_{i=1}^{n}\frac{a}{2a-a_i}\le\sum_{i=1}^{n}\frac{na_i}{2a-a_i}.$$

Since

$$\frac{a}{2a-a_i}=\frac{1}{2}+\frac{1}{2}\cdot\frac{a_i}{2a-a_i},$$

we get

$$\frac{n}{2}+\frac{1}{2}\sum_{i=1}^{n}\frac{a_i}{2a-a_i}\le n\sum_{i=1}^{n}\frac{a_i}{2a-a_i}.$$

From here we obtain the desired inequality.
Equality holds iff $a_1=a_2=...=a_n=1$.

Example 9. If $a,b,c>0$, prove tha

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\ge\frac{a^2+b^2+c^2}{2}.$$

Solution: By symmetry we may assume that $a\le b\le c$. Then $a+b\le c+a\le b+c$. So, we have

$$\frac{1}{b+c}\le\frac{1}{c+a}\le\frac{1}{a+b}.$$

By the rearrangement inequality we have

$$\frac{a^3}{a+b}+\frac{b^3}{b+c}+\frac{c^3}{c+a}\le\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b},$$

$$\frac{a^3}{c+a}+\frac{b^3}{a+b}+\frac{c^3}{b+c}\le\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}.$$

Adding these inequalities and then dividing by 2, we get

$$\frac{1}{2}\left(\frac{a^3+b^3}{a+b}+\frac{b^3+c^3}{b+c}+\frac{c^3+a^3}{c+a}\right)\le\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}.$$

Finally, since

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2\ge\frac{x^2+y^2}{2},$$

we have

$$\frac{a^2+b^2+c^2}{2}=\frac{1}{2}\left(\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}\right)\le\frac{1}{2}\left(\frac{a^3+b^3}{a+b}+\frac{b^3+c^3}{b+c}+\frac{c^3+a^3}{c+a}\right)\le\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b},$$ i.e.

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq\frac{a^2+b^2+c^2}{2}\text{, q.e.d.}$$

Equality holds true iff $a=b=c$.

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