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Научно-методически статии

THE BENEFITS OF COLLABORATIVE LEARNING IN EXAMINING FUNCTIONS WITH PARAMETERS IN DYNAMIC SOFTWARE ENVIRONMENT

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Abstract. In this paper the efficiency of collaborative learning of functions, of the functions with parameters, in *GeoGebra* environment is analyzed. The research was conducted during the calculus course at the University of Novi Sad, Serbia, with the students with two groups of students, major physics and chemistry. The students in the experimental group learned in small, four member groups, formed by using Kagan's (1994) principles, and the students in the control group learned individually. The students from both groups learned in *GeoGebra* environment. Their work is compared and analyzed.

Keywords: calculus; collaborative learning; dynamic software; functions with parameters

1. Introduction

Numerous researches showed that the students have a lot of difficulties with calculus contents within their Mathematics courses. At the beginning stages of calculus, definitions (functions, limits, continuity and integrals) and their applications are given rigid, and this is very difficult for students to understand. In particular, examining the properties of functions, and their multiple representations, algebraic and graphical turned out to be one of most difficult students' tasks. Since, the functions are used in a lot of students' further courses and their professional work it is very important for students to overcome the difficulties in understanding the function concept (Borba & Confrey 1996; Daher & Anabousy 2015; Tall 1992; Takači, Stankov & Milanović 2015).

In the last two decades, different dynamic packages, as *Mathematica*, *GeoGebra*, *Maple*, *Cinderella*, are applied in teaching and learning calculus contents. All mentioned dynamic software is appropriate for examining the families of functions, algebraically represented as the functions with parameters. They enable easy change of variables, providing simultaneous dynamic multiple representations of families of functions and their properties (Anabousi, Daher, Baya'a & Abu-Naja 2014; Borba & Confrey 1996; Daher & Anabous 2015).

All students, included in the research presented in this paper, worked in Geogebra environment, i.e., all of them have the same possibilities to use its dynamic properties within multiple representations of functions with parameters. The experimental group worked in small collaborative groups and the students collaborative learning, in *GeoGebra* environment, (discussions and ideas exchange), in order to successfully solve the given tasks related to the functions with parameters is analyzed and compared with the control groups of students who worked the same tasks individually.

2. Theoretical background

2.1. Collaborative learning

One of the teaching methods with special importance in modern education, and which we applied in our research, is collaborative learning. In modern learning theory, Vygotsky is among the first to emphasize the importance of the collaboration between peers during the learning process (Vygotsky 1978).

Collaborative learning also contributes to training students for teamwork. The process of problem solving is very complex, and in order to be successful, it is necessary that the team works well and that cooperation among the members of the group is as successful as possible. In addition, when learning in a group, each member of the group bears responsibility, not only his own, but also for other members of the group. This significantly contributes to the development of critical thinking, cooperation, responsibility towards associates and to working together (Gokhale 1995; Laal & Ghodsi 2012).

The application of modern technology has largely contributed to the development of collaborative learning and its successful application in teaching. Using computers enabled the creation of an approach known as computer supported collaborative learning. Under this approach, there are different combinations of collaborative learning and learning by computer (Miyake 2007). In some cases, the computer is used as an auxiliary teaching tool in the work of a collaborative group. On the other hand, computers, as well as smart phones, are increasingly used as communication tools between members within a collaborative group. Computers are also often used in order to improve scripts and teaching material (Weinberger & Fischer 2006).

Collaborative groups can be formed in different ways. Kagan (1994) described a method of creating four member heterogeneous groups, which proved, by his experience, to be very efficient. Groups should be formed taking into account the students different knowledge and their feelings towards one another (i.e. students' wishes to work or not to work together). Firstly, students' previous knowledge should be evaluated and, in accordance with this evaluation, rank list of the students should be formed. This method is also used in the research Takači et al. (2015), where the procedure of forming groups is described in detail: "The first group should be formed by taking the first student, the last one and two students from the middle of the rank list. If there are no negative feelings and close friendships among these four students, then the group is accepted. In the opposite case, the person from the middle of the rank list is replaced with the nearest one. If the first group is formed properly, its members are omitted from the list and the process is repeated." (Takači et al. 2015)

2.2. Multiple representations

Representations in mathematics, as representations at all, are usually considered "as a process of modeling concrete objects in the real world into abstract concepts" (Hwang & Hu 2013). The observer puts two concepts one against the other, revealing and comparing the similarities and differences between them. In this way one of the observed concepts is represented (Font, Godino & D' Amore 2007). Using one system to represent another is usual, even for young children (Lehrer & Schauble 2000). The classification of representations was the subjects of many researches. Usually two large groups of representations are considered – internal representation, created in the mind of an individual, and external, created in his surroundings (Goldin & Shteingold 2001; Kilpatrick, Swafford & Findell 2001; Nakahara 2008).

The advantage of using computers in the formation of multiple representations is especially evident when connecting different representations. Namely, there are software packages that allow simultaneous display of two to three representations of the same object. More recently, dynamic software packages which enable forming the multiple representations are especially important. These software packages enable dynamically linking the multiple representations and creating the system of multiple representations, where a change in one representation causes simultaneous change in other representations of the same object (Hwang & Hu 2013). Dynamic software also enables forming of so-called virtual manipulative representation. This representation is an interactive visual representation of a dynamic object, which allows students to manipulate objects, i.e., to change one property of the object and at the same time to observe how the change of that property affects the other properties of the object.

2.3. Multiple representations of functions

Functions are mathematical concept that can be presented in many ways. One representation is usually not sufficient to represent all the properties of a function. Therefore, multiple representations are usually used to display the functions' properties. Usually, the most important is the algebraic, graphical and tabular representation, but, in recent times, more and more verbal representation is gaining importance. The use of the computers brought innovations in a representation of functions. Multiple representations of functions in a computer based environment have been topic of many papers (Goldin & Kaput 1996; Borba & Confrey 1996; Doorman, Drijvers, Gravemeijer, Boon & Reed 2012).

Transformations are connected with parameters. They can be considered as a consequence of the parameter's value changing. The use of multiple representations helps students to analyze properties of the functions from different points of view (Anabousy, Daher, Baya'a & Abu-Naja 2014; Borba & Confrey 1996; Daher & Anabousy 2015).

The students have to work with different representations, because each representation enables an adequate overview of some properties of the functions, but no representation provides a complete overview of functions' properties (Doorman et al. 2012). By analyzing connections between different representations, students can note some characteristics which they wouldn't note by considering each representation separately, because they can observe the dependence between the properties of functions. By observing different representations simultaneously, the students are enabled to choose the most appropriate representation for each case separately (Borba & Confrey 1996).

GeoGebra is one of the software packages which enables connecting different representations and work within multiple representations of functions. It also enables forming the dynamic multiple representations of the functions, which are being formed by creating sliders, by which variable parameters are defined. The moving of the slider causes immediate changes of the parameter's value and, consequently, causes changes in algebraic and graphical representation, simultaneously.

3. Research question

The aim of this research is to point out how students, working in collaborative groups, in *GeoGebra* environment, exchange their opinions and make joint decisions and conclusions about the properties of the observed functions with parameters. Due to the aims of the research, the main research questions are:

> How do students manage their collaborative learning, in *GeoGebra* environment, (discussions and ideas exchange), in order to successfully solve the given tasks related to the functions with parameters?

4. Methodology

4.1. General background

In this research, experimental approach was applied and the experiment is conducted with parallel groups: the experimental and the control group. In the experimental group, collaborative learning was applied, and in the control group students worked individually. The benefits of work within collaborative groups are examined.

All students in the experimental group worked in the four member heterogeneous collaborative groups. These collaborative groups were formed by using Kagan's instructions (Kagan 1994), in the manner applied in research Takači et al. (2015).

4.2. Participants

The research is conducted with 120 undergraduate students, during their calculus course, at Faculty of Sciences, University of Novi Sad, Serbia, in 2017.

4.3. Instruments and procedures

Before the research was conducted, all students solved a pre-test. The rank list of students, based on the pre-test results, is created. By taking into account these results, the experimental and control group are formed (with 60 students in each group), so the difference between these groups was not statistically significant at the level of significance 0.01.

The collaborative groups, within experimental group, were formed on the basis of pre-test results and students' feelings towards one another (likes/dislikes each other). All groups consist of four students each. The first group was formed by taking the first student, the last one, and two students from the middle of the rank list. There were no negative feelings and close friendships among these four students, so the group is accepted. Then, members of the first group were omitted from the list and the process was repeated. When it happened that there were negative feelings or close friendships among students in the formed group, then one student was replaced with the nearest one on the rank-list. In that way, 15 four-member collaborative groups are formed.

After forming the experimental and the control group, students' work on examining properties of families of functions is conducted. The students solved the tasks at the university, and this process lasted about 7 hours (2 days, 3 hours on the first and 4 hours on the second day).

It is important to note that students, before the exercises covered by this research, were introduced to functions and their properties, as well as the properties of their derivatives, and that they had already worked within the dynamic environment of the *GeoGebra* software package. This research includes the practice of examining the properties of functions with parameters and related functions families.

Before their exercises, the students were introduced to procedure of the learning process and the students of the experimental group were informed to which group they belong to. During task-solving, all students, regardless of whether they belonged to the experimental or the control group, had possibilities to use *GeoGebra* software and the Internet. The learning process of both groups was monitored by teachers, who (if needed) gave some necessary instructions (the explanations of the tasks requirements, as well as the use of *GeoGebra*). These instructions could be given to the students of the control group, who worked individually, as well as the students of the experimental group, who worked in small collaborative groups. The teachers reviewed students' works and all students got 5 points for successfully completed tasks.

5. The analysis of students' work

5.1. Students' collaborative work

The students in the experimental group made the activity plan and split their roles within their own groups before they started to work. When they got their tasks (Appendix), they started work regarding their plans. In 9 groups the students brought their laptops and one student began to work with the computer, while 3 other students (from the small group) analyzed the tasks, suggesting the ways for solving tasks. In the remaining 6 groups, all students have started to use their own smart phones, with *GeoGebra* package already installed. After a while, one student in each group has started to lead and to direct the activities in order to obtain better multiple representations of the functions given with parameters.

All students, in the experimental group, were interested in their collaborations and actively worked together. Their challenge was to organize multiple representations of functions as best as they can, in order to examine the properties of functions and their dependence on parameters. All members of the group, jointly, by discussion, made conclusions about the properties of the given families of functions and the influence of parameters on these properties, using *GeoGebra* dynamic environment. The students often had to change the original work plan of the group, so each member was equally involved in the observation of the resulting changes in the dynamic worksheets, and the conclusions were made jointly, by discussion. In all small groups, the students were adjusting their roles to the requirements of the tasks. The students in charge of computer work included themselves in discussions and they were replaced from time to time, so that each member of the group takes on some of these duties.

Mostly, all small groups worked with one computer, or smart phone. In the groups where smart phones were used (at the beginning) it happened that (later) only one phone was chosen for the further analysis. The students concluded that it is more convenient to follow the discussions about the differences and similarities within a family of functions, multiple represented, if everything is done on only one display.

Since they were supervised by the teacher participating in students' discussions, they were suggested to use sliders (with integer values) in order to represent functions with parameters. Firstly, they were changing the values of one parameter by moving the slider, and for each value of the parameter they obtained simultaneous algebraic and graphical representation of the function and its derivatives. By observing the students' work, it is noticed that the students were very satisfied working with *GeoGebra* package, because it enables all tools necessary for examining functions. In the second part of the first task, the students had to change the two parameters. After a short discussion, they introduced the second slider, for that purpose. Then they fixed a value of one parameter and, by changing the other parameter, they got the same situation as in the first part of the task. As it was expected, it was more difficult, but all students worked on this task very successfully.

The teacher remarked that the students showed a great deal of variety in their work within the small groups and their discussions were very different. Also, there were no two identical worksheets, even tasks were the same and the solutions were correct. They used different possibilities of *GeoGebra* package and got interesting algebraic and graphical representations of the families of functions. For example, the easiest part was the examining the influence of parameters a and b on the properties of family of functions $f(x+a)+b$, because parameter b does not affect the properties of the first derivative and corresponding properties of functions (monotonicity). In Figure 1, students' dynamic worksheet of the 4th group for the families of functions $f(x+a)+b$, where *f* is a rational function, is shown. The students fixed value of parameter $a=1.6$ and used the command "trace" in order to obtain different graphical representations of functions, depending on the value of parameter b. The function p is represented both, algebraically and graphically, for value of parameter $b=-5$. The derivative is the same, given algebraically and graphically in "Graphics 2" view. Also, within this worksheet, the students used different colors and style of lines.

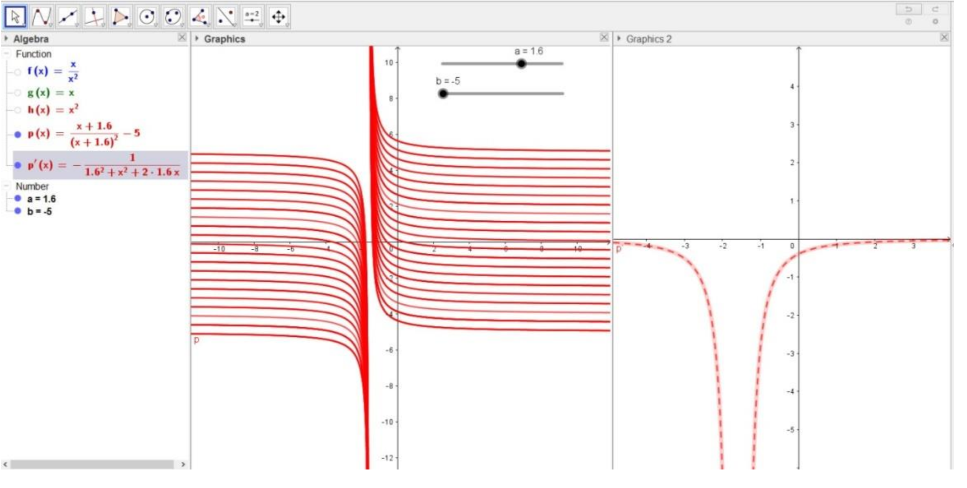

Figure 1. Work of the 4th group of students

The students' worksheet of the 12^{th} group (Figure 2) is interesting, because one function from the family $f(ax)+b=\frac{g(ax)}{h(ax)}+b$ and one function from the family $\frac{g(ax)+b}{h(ax)+b}$, which were analyzed and compared, are shown in Graphic 2 view, together with their derivatives.

The students' worksheet of the 7^{th} group (Figure 3) is interesting, because the students used Graphic 2 view for presenting the properties of functions. This worksheet is prepared by using dynamic possibilities in presenting properties of functions, such that each reader can change the values of parameters and follow the change of their properties.

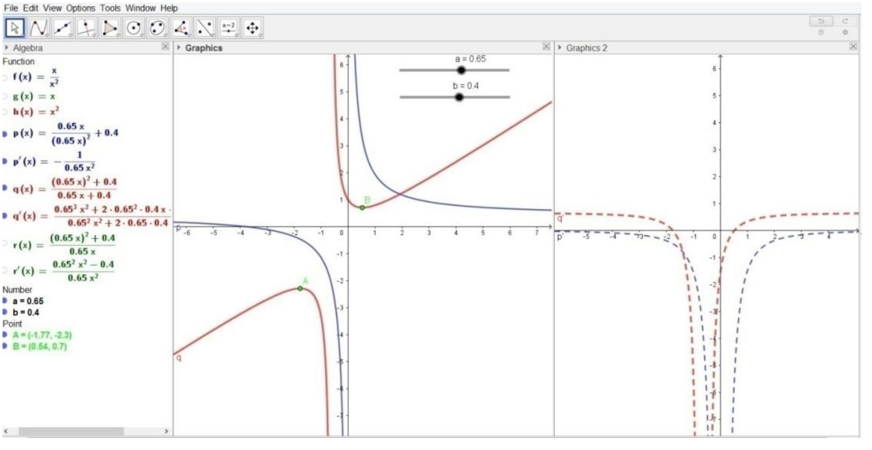

Figure 2. Work of the 12^{th} group of students

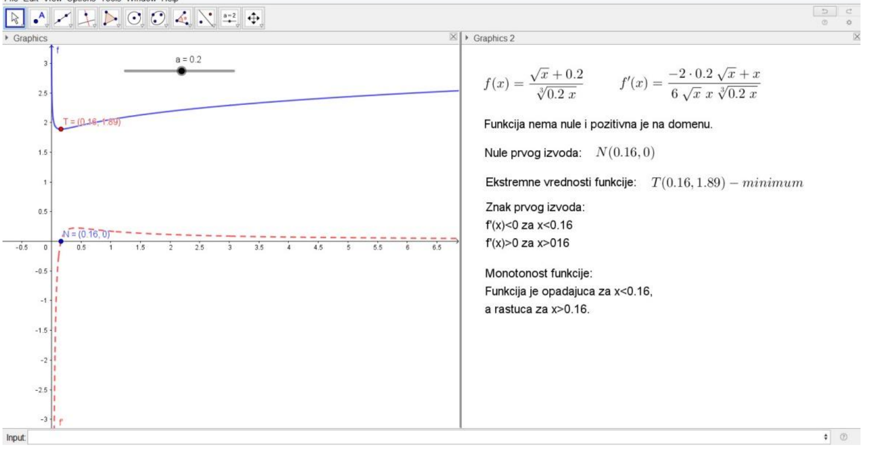

Figure 3. Work of the 7^{th} group of students

5.2. Analysis of students' solutions of the tasks

The teacher did not suggest the form of the finished students' tasks, and they could prepare it by hand, by computer only, or by using their combinations.

Each collaborative group has prepared the final solution of tasks, differently. There were 8 groups (32 students) whose final solutions were prepared as the combination of hand and computer applications and 7 groups (28 students) whose tasks were solved by computer only (Table 1). All groups who used their smart phones (6 groups) had their final solution in the combination of both applications, while 7 groups working with big computers had only computer applications of solutions. Nobody in the experimental group had only hand written solutions, while in the control group there were 29 such solutions (Table 1). A possible reason for this is inability of control group students to split roles, i.e. every student of the control group had to work within *GeoGebra* on the computer or smart phone (39 students of the control group used big computers and the rest 21 used smart phones) and to write solution. So, maybe for this reason, many of them decided to create a final solution only in paper form.

Table 1. *The way of creating students' attached materials*

Way of creating attached material	Computer	Hand write	Combination (computer and hand)
Number of students (experimental group)	28	0	32
Number of students (control group)	6	29	25

It is interesting to note that students, working in collaborative groups, made progress in preparing final solutions, by using computers only or in combination with written documents, while in control group even 48% of students used classical handwritten document.

For example, the tasks of the 4th and 12th small group (Figures 1 and 2) were prepared as the combination of both, written work and computer applications, while the 7th small group prepared all worksheets only by computer. One of them is shown in Figure 3.

GeoGebra environment enables examining functions by using only graphical representations and geometric notions, such as the corresponding intersection of objects. For example, in the experimental group, the 4th group (Figure 1) used graphical representation, obtained by using the command "trace". In control group 11 students (18%) used only graphical representations, while in the experimental group almost all students used both, algebraic and graphical representation. In Table 2, number of students (groups), which used graphical and both representations, is shown. Nobody used only algebraic representations for examining families of functions.

Table 2. *The use of algebraic and graphical representation by small groups/ students*

The way of creating attached material	Computer		Hand		Both	
Group \ Representation used	Both	Graphical	Both	Graphical	Both	Graphical
Experimental	6 groups	1 group	0	0	8 groups	0
Control	4	2	29	0	16	9

Most of the students in experimental group considered all properties of families of functions and their dependence on the parameters, but some of them, as well as the students from the control group, omitted some of them. Below the most interesting parts of works, from mathematical point of view, are presented. An overview of the number of students who correctly explained the influence of parameters on certain property of the function is given in Table 3.

Domain: In the experimental group 12 small groups, i.e., 48 students discussed about the influence of parameter on the domain of the families of functions while in the control group only 34 students did that.

Period: Almost all students (except one group) considered the periodicity of trigonometric functions in determining the influences of parameters on the properties of functions, while in the control group only 37 students did it correctly.

Zeros: All students, in both the groups, discussed about successfully, but students in the experimental group were better with periodic function.

Asymptotes: In the experimental group there were 6 small groups who analyzed vertical asymptotes of logarithmic function and horizontal asymptotes of exponential functions and their dependence on parameters, within corresponding family. In the control group, only 2 students analyzed these properties.

Derivatives: All students, working with *GeoGebra*, obtained the derivatives by using appropriate commands and they had to connect them and their properties with the functions' properties. Sometimes, the algebraic representations were very complicated for the analyses and the students used commands CAS in order to make it more convenient for comparing properties, or they used their graphical representations.

Monotonicity and extreme values: It is interesting to note there were 5 groups (20 students) from the experimental group who used only graphical representations of functions to determine the monotonicity and the extreme values, working with rational, exponential and logarithmic functions. They did not take into account the connection with derivatives. In the control group only 3 students connected the properties of derivatives with the properties of functions.

Table 3. *Number of the students who correctly explained the influence of parameters on certain property of the function*

Property of function \ Group	Exp	Control
Domain	48	34
Period	56	37
Zeros	60	54
Asymptotes	24	2
Derivatives, Mononicity, Extremes	44	33

Appendix. *The tasks solved by the students during their work on examining properties of types of functions with parameters*

1. For different values of the parameters a and b, sketch the graphs and examine the properties of next functions and their derivatives:

$f(x)+a$, $f(x+a)$, $af(x)$ и $f(ax)$,

$f(x+a)+b$, $af(x)+b$, $f(ax)+b$ и $f(ax+b)$,

where the functions f are given in the table below. Examine the influence of parameters a and b on the properties of functions (zeros, sign, monotonicity), as well as on the properties of the first derivative.

A)	$g(x)=x$	$h(x)=x^2$	$f(x)=\dfrac{g(x)}{h(x)}$	
B)	$f(x)=2^x$	$f(x)=\left(\dfrac{1}{3}\right)^x$	$f(x)=1.5^x$	$f(x)=0.4^x$
C)	$f(x)=\log_2 x$	$f(x)=\log_{1/3} x$	$f(x)=\ln x$	$f(x)=\log_{0.4} x$
D)	$f(x)=\sin x$	$f(x)=\cos x$	$f(x)=\operatorname{tg} x$	$f(x)=\operatorname{ctg} x$
E)	$f(x)=\sqrt{x}$	$f(x)=\sqrt[3]{x}$	$f(x)=\dfrac{1}{\sqrt{x}}$	$f(x)=\dfrac{1}{\sqrt[3]{x}}$

Examine the properties and sketch the graph of the function $f(x)=\frac{x^2-a}{x-b}e^{cx}$, depending on the values of parameters a, b and c.

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