Educational Technologies
Образователни технологии

SOME NUMERICAL SEQUENCES CONCERNING SQUARE ROOTS (PART TWO)

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Abstract. The present article is a continuation of the homonymous publication in the journal "Mathematics and Informatics", issue 5, 2017. The theoretical considerations in it cover all cases when calculating the value of a finite and infinite number of nested square radicals.

Keywords: infinite sequence; nested radical; convergence; mathematical induction

In the first part of this research we examined the convergence of the numerical series $c_n = \sqrt{a + bc_{n-1}}$, $c_1 = \sqrt{a}$, where $a > 0$ and $b > 0$. The aim of the second part is to determine conditions for the limit existence including the case 0.

For a starting base and background we will use again some basic skills and knowledge in the field of:

Infinite numerical series;

Convergence of infinite numerical series;

Limits of infinite numerical series.

On this theoretical basis we will apply some specific and non-standard skills for proving scientifically determined dependencies.

Using theorem 1 from Part one (Nikolaev, Milkova & Petkov, 2017) for example we can easily find the value of the expression $L = \sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + \dots}}}$.

Concerning this, a question arises – can we solve the next problem in a similar way?

Determine the value of the expression $L = \sqrt{3 - 2\sqrt{3 - 2\sqrt{3 - \dots}}}$.

The answer seems to be positive and the solution is based on the statements mentioned in the present paper.

We know from theorem 1 that if $t > b > 0$, then $\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{\dots}}} = t$.

We will formulate and prove the following theorem:

Theorem 2. If $b<0$ and $L>\frac{b\left(1-\sqrt{5}\right)}{2}$, then $\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\ldots}}}=L.$

It is obvious that $L>0.$

We examine the numerical series $c_n=\sqrt{L\left(L-b\right)+bc_{n-1}}$, with its first element $c_1=\sqrt{L(L-b)}$.

This numerical series can be divided into two subseries – the first one consisting of the odd elements $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ and the other one ofh the even elements, i.e. $\left\{c_{2k}\right\}_{k=1}^{\infty}$.

We will formulate and prove two more theorems which will help in proving Theorem 2.

Theorem 3. The subseries $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ converges and when b<0 its limit is equal to $L>\frac{b\left(1-\sqrt{5}\right)}{2}$.

Theorem 4. The subseries $\left\{c_{2k}\right\}_{k=1}^{\infty}$ converges and when $b<0$ its limit is equal to $L>\frac{b\left(1-\sqrt{5}\right)}{2}$.

To prove theorem 3 in an easier way we will formulate three lemmas.

Lemma 1.The subseries $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ is limited from below.

Proof: We use the principle of mathematical induction.

1. $k=1, c_1=\sqrt{L(L-b)}$.

We will prove that $\sqrt{L(L-b)}>L.$

$\sqrt{L(L-b)}>L\,\Big|^2;$

$L(L-b)>L^2;$

$-Lb>0,$ since $L>0, b<0$ (by condition)

$\Rightarrow \mathrm{c}_1>L$ - correct (1)

$c_3=\sqrt{L(L-b)+bc_2}$

We will prove that $c_3=\sqrt{L(L-b)+bc_2}>L.$

$\sqrt{L(L-b)+bc_2}>L\,\Big|^2;$

$L(L-b)+bc_2>L^2;$

$bc_2>Lb\,\Big|:b<0;$

$c_2<L;$

$c_2=\sqrt{L(L-b)+bc_1}.$

Let us check if $c_2 < L$:

$c_2 = \sqrt{L(L-b)+bc_1} < L \Leftrightarrow L(L-b)+bc_1 < L^2 \Leftrightarrow bc_1 < Lb.$

Since $b < 0$, then $c_1 > L$ - correct, concluded from (1).

2. Let us assume that for each $k = 1, 2 \ldots n$ it is true that $c_{2k-1} > L.$

3. We will prove that $k = ùn +$ $c_{2(n-+1)\ 1} = c_{2n\ 1} > L$

If $c_{2n+1} = \sqrt{L(L-b)+bc_{2n}} > L,$

$c_{2n+1} = \sqrt{L(L-b)+bc_{2n}} > L \,\Big|^2 \Leftrightarrow L^2 - Lb + bc_{2n} > L^2 \Leftrightarrow bc_{2n} > Lb \,\Big|: b<0 \Leftrightarrow c_{2n} < L;$

$c_{2n} = \sqrt{L(L-b)+bc_{2n-1}} < L \,\Big|^2 \Leftrightarrow L^2 - Lb + bc_{2n-1} < L^2 \Leftrightarrow bc_{2n-1} < Lb \,\Big|: b<0 \Leftrightarrow c_{2n-1} > L$,

which is correct according to the assumption in point 2.

In such a way we proved that the subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is limited from below, i.e. $c_{2k-1} > L.$

Lemma 2. The subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is decreasing monotonously.

Proof: Once again we will use the principle of mathematical induction.

1. $k = 1.$

We will check whether $c_1 > c_3.$

$\sqrt{L(L-b)} > \sqrt{L(L-b)+bc_2} \Leftrightarrow L(L-b) > L(L-b)+bc_2 \Leftrightarrow 0 > bc_2.$

Since $b < 0$, then c_2 must be positive.

$c_2 = \sqrt{L(L-b)+bc_1}$, and for the value under the radical to be defined we need $L(l-b) + b\sqrt{L(L-b)} > 0$.

$L(l-b) > -b\sqrt{L(L-b)} \,\Big|^2 \quad -b > 0,\ L(L-b) > 0;$

$\left[L(L-b)\right]^2 > b^2 L(L-b) \,\Big|: L(L-b) > 0;$

$L(L-b) > b^2;$

$L^2 - Lb - b^2 > 0 \quad \Leftrightarrow \quad L \in \left(-\infty; L_1\right) \cup \left(L_2; +\infty\right),$

where L_1 and L_2 are the roots of the equation $L^2 - Lb - b^2 = 0$:

$$L_{1,2} = \frac{b \pm \sqrt{b^2 + 4b^2}}{2} = \frac{b\left(1 \pm \sqrt{5}\right)}{2};$$

$$L_1 = \frac{b\left(1+\sqrt{5}\right)}{2} < 0; \quad L_2 = \frac{b\left(1-\sqrt{5}\right)}{2} > 0$$

$$\Rightarrow L \in \left(-\infty; \frac{b\left(1+\sqrt{5}\right)}{2}\right) \cup \left(\frac{b\left(1-\sqrt{5}\right)}{2}; +\infty\right).$$

In theorem 2 we stated that $L > \frac{b\left(1-\sqrt{5}\right)}{2}$, and in such a way we can make the conclusion that $c_1 > c_3$.

2. We assume that $c_{2k-1} > c_{2k+1}$, for each $k = 1,2\ldots n$.

3. We will prove that this is satisfied also for $k = 2n+1$: $c_{2n+1} > c_{2n+3}$.

$$c_{2n+3} = \sqrt{L(L-b)+bc_{2n+2}} < c_{2n+1} \Leftrightarrow L(L-b)+bc_{2n+2} < c^2_{2n+1} \Leftrightarrow$$

$$\Leftrightarrow \cancel{L(L-b)} + b\sqrt{L(L-b)+bc_{2n+1}} < \cancel{L(L-b)} + bc_n \; \Big|: b<0$$

$$\Leftrightarrow \sqrt{L(L-b)+bc_{2n+1}} > c_n = \sqrt{L(L-b)+bc_{2n-1}}$$

$$\Leftrightarrow bc_{2n+1} > bc_{2n-1} \; \Big|: b<0$$

$$\Leftrightarrow c_{2n+1} < c_{2n-1},$$

which is correct according to the assumption of the induction.

This way we proved that the subseries $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ is decreasing monotonously.

Lemma 3. The limit of the series $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ is equal to *L*.

Proof: From Lemma 1 and Lemma 2 we can conclude that $\left\{c_{2k-1}\right\}_{k=1}^{\infty}$ has a limit denoting it by *l*. Then:

$\lim\limits_{k\to\infty} c_{2k-1} = l$ and $\lim\limits_{k\to\infty} c_{2k+1} = l$.

From the recurrence

$c_{2k+1} = \sqrt{L(L-b)+bc_{2k}} = \sqrt{L(L-b)+b\sqrt{L(L-b)+bc_{2k-1}}}$, after a limit transition, we receive an irrational equation for *l*:

$$l = \sqrt{L(L-b)+b\sqrt{L(L-b)+bl}} \Leftrightarrow l^2 - L(L-b) = b\sqrt{L(L-b)+bl} \Leftrightarrow$$

$$\left|\begin{array}{l} l^2 - L(L-b) \le 0 \\ L(L-b)+bl \ge 0 \\ \left[l^2 - L(L-b)\right]^2 = b^2(L(L-b)+bl) \end{array}\right. \Leftrightarrow \left|\begin{array}{l} l \in \left[-\sqrt{L(L-b)}; \sqrt{L(L-b)}\right] \\ l \in \left(-\infty; \frac{L(L-b)}{-b}\right] \\ \left[l^2 - L(L-b)\right]^2 = b^2(L(L-b)+bl) \end{array}\right. .$$

If we consider from Theorem 2 that $L > \frac{b\left(1-\sqrt{5}\right)}{2} > 0$ together with Lemma 1 then the limit *l* will be positive. After some transformation, the last system turns to be equivalent to:

$$\begin{cases} l \in \left(0; \sqrt{L(L-b)}\right] \\ l^4 - 2(L^2 - Lb)l^2 - b^3 l + (L^2 - Lb)(L^2 - Lb - b^2) = 0 \end{cases}$$

Then we obtain immediately that $l_1 = L$ and $l_2 = b - L$ are roots of the above mentioned equation. After factorising its left hand side we get the equivalent form:

$(l - L)(l - b + L)(l^2 + bl + Lb - L^2 + b^2) = 0$,

and consequently $l_3 = \dfrac{-b - \sqrt{4L^2 - 3b^2 - 4Lb}}{2}$ and $l_4 = \dfrac{-b + \sqrt{4L^2 - 3b^2 - 4Lb}}{2}$.

But from $b < 0$ and $L > \dfrac{b\left(1 - \sqrt{5}\right)}{2} > 0$, we have

$l_1 = L \in \left(0; \sqrt{L(L-b)}\right]$, $l_2 = b - L < 0$, $l_3 = \dfrac{-b - \sqrt{4L^2 - 3b^2 - 4Lb}}{2} < 0$

and $l_4 = \dfrac{-b + \sqrt{4L^2 - 3b^2 - 4Lb}}{2} > \sqrt{L(L-b)}$. Hence $l = L$.

So, from the proved Lemma 1, Lemma 2 and Lemma 3 in advance we conclude that thr assertion in Theorem 3 is true.

For proving Theorem 4 we will formulate three more lemmas.

Lemma 4. The subseries $\left\{c_{2k}\right\}_{k=1}^{\infty}$ is limited from above.

Proof: We will use again the principle of mathematical induction.

1. $k = 1$, $c_2 = \sqrt{L(L-b) + bc_1}$.

We will prove that $c_2 = \sqrt{L(L-b) + bc_1} < L.$

$\sqrt{L(L-b) + bc_1} < L.\Big|^2$;

$L^2 - Lb + bc_1 < L^2$;

$Lb > bc_1$, since $L > 0, b < 0$ (by condition)

$\Rightarrow c_1 > L$ - true to Lemma 1.

If $k = 2$, $c_4 = \sqrt{L(L-b) + bc_3}$.

We will prove that $c_4 = \sqrt{L(L-b) + bc_3} < L$

$\sqrt{L(L-b) + bc_3} < L\Big|^2$

$L^2 - Lb + bc_3 < L^2$

$bc_3 < Lb \;\Big|: b < 0$

$c_3 > L$ – true to Lemma 1.

2. We assume that for each $k = 1, 2 \ldots n$ it is true that $c_{2k} < L$.

3. We will prove that if $k = n+1$ $\; c_{2(n+1)} = c_{2n+2} < L$.

And if $c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} < L$,

$c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} < L \,\Big|^2 \Leftrightarrow L^2 - Lb + bc_{2n+1} < L^2 \Leftrightarrow bc_{2n+1} < Lb \,\Big|: b < 0 \Leftrightarrow c_{2n+1} > L$, which can be concluded by Lemma 1.

So we proved that the subseries $\{c_{2k}\}_{k=1}^{\infty}$ is limited from above by the constant L, i.e. $c_{2k} < L$.

Lemma 5. The subseries $\{c_{2k}\}_{k=1}^{\infty}$ is increasing monotonously.

Proof: We will use once again the principle of mathematical induction.

1. $k = 1$.

We will check whether $c_4 > c_2$.

$\sqrt{L(L-b) + bc_3} > \sqrt{L(L-b) + bc_1} \Leftrightarrow L(L-b) + bc_3 > L(L-b) + bc_1 \Leftrightarrow bc_3 > bc_1$.

Because of $b < 0$, we have $c_3 < c_1$, which can be concluded by Lemma 2.

2. We assume that $c_{\grave{u}k} > 2\ _{k-}$, for each $k = 1, 2 \ldots n$.

3. We will prove that this is also correct for $k = n+1$: $c_{2n+2} > c_{2n}$.

$$c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} > c_{2n} \Leftrightarrow L(L-b) + bc_{2n+1} > c_{2n}^2 = L(L-b) + bc_{2n-1}$$

$$\Leftrightarrow bc_{2n+1} > bc_{2n-1} \,\Big|: b < 0$$

$$\Leftrightarrow c_{2n+1} < c_{2n-1},$$

which is a consequence of Lemma 2.

Hence, Lemma 5 is proved.

Lemma 6. The limit of the series $\{c_{2k}\}_{k=1}^{\infty}$ is equal to L.

Proof: From Lemma 4 and Lemma 5 we can make the conclusion that the subseries $\{c_{2k}\}_{k=1}^{\infty}$ has a limit and denote it by l. Then:

$$\lim_{k \to \infty} c_{2k} = l.$$

From the recurrence

$c_{2k} = \sqrt{L(L-b) + bc_{2k-1}}$, after a limit transition and applying Theorem 3, we can conclude that

$$l = \lim_{k \to \infty} c_{2k} = \lim_{k \to \infty} \sqrt{L(L-b) + bc_{2k-1}} = \lim_{k \to \infty} \sqrt{L(L-b) + bL} = L.$$

Thus, $l = L$.

Theorem 4 follows by Lemma 4, Lemma 5 and Lemma 6.

Coombining Theorem 3 and Theorem 4 we receive the statement of Theorem 2.

Following the definition of the square root, the following conclusions could be made from Theorem 2:

Conclusion 1: If $L(L-b)<0$, then the expression $\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}}$ is not defined in the set of the real numbers.

Proof: $c_1=\sqrt{L(L-b)}$ does not exist because $L(L-b)<0$.

The following conclusion is also obvious.

Conclusion 2: If b < 0 and $L(L-b)<0$, then the expression

$$\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}}$$

is not defined in the set of the real numbers.

Conclusion 3: If $b<0$ and $L<\frac{b\left(1-\sqrt{5}\right)}{2}$, then the expression

$$\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}}$$

is not defined in the set of the real numbers.

Proof: The elements of the subseries $\left\{c_{2k}\right\}_{k=1}^{\infty}$ have negative values under the square root.

Theorem 5. Let $b<0$ and $L=\frac{b\left(1-\sqrt{5}\right)}{2}$. Then

$$\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}}=\begin{cases}-b, \text{ if the number of roots is a finite odd number,}\\ 0, \text{ if the number of roots is a finite even number,}\\ \textit{indefinite}, \text{ if the number of roots is infinite.}\end{cases}$$

Proof: We will use again the principle of mathematical induction.

1. $n=1$.

$$c_1=\sqrt{L(L-b)}=\sqrt{\frac{b\left(1-\sqrt{5}\right)}{2}\left(\frac{b\left(1-\sqrt{5}\right)}{2}-b\right)}=\sqrt{\frac{b\left(1-\sqrt{5}\right)}{2}\left(\frac{b\left(-1-\sqrt{5}\right)}{2}\right)}=\sqrt{b^2}=|b|;$$

$b<0\Rightarrow c_1=-b.$

$c_2=\sqrt{L(L-b)+bc_1}=\sqrt{L(L-b)+b(-b)}=\sqrt{L(L-b)-b^2}=\sqrt{b^2-b^2}=0.$

2. Let us assume that $c_{2n-1}=-b$ and $c_{2n}=0$ for each $n=1,2\dots k$.

3. We will prove that for $n=k+1$ it is true that $c_{2k+1}=-b$ and $c_{2k+2}=0$.

$c_{2k+1}=\sqrt{L(L-b)+bc_{2k}}=\sqrt{b^2+b.0}=\sqrt{b^2}=|b|=-b.$

$c_{2k+2}=\sqrt{L(L-b)+bc_{2k+1}}=\sqrt{b^2-b^2}=0.$

If the number of the roots is infinite, then the value of the expression $\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\ldots}}}$ cannot be determined.

The proved statements complete the possibilities to calculate expressions of the type $\sqrt{a+b\sqrt{a+b\sqrt{\ldots}}}$.

Applying the proved statements, we will examine three problems.

Problem 1. Determine the value of the expression $L=\sqrt{3-2\sqrt{3-2\sqrt{3-\ldots}}}$.

Solution:

$$b=-2,\ a=L(L-b)=L(L+2)=3;$$

$$L^2+2L-3=0;$$

$$L_1=-3,\ L_2=1;$$

$$L=\frac{b\left(1-\sqrt{5}\right)}{2}=\frac{-\not{2}\left(1-\sqrt{5}\right)}{\not{2}}=\sqrt{5}-1.$$

We compare $L_1=-3$ and $L_2=1$ with $\sqrt{5}-1$.

Since $b<0$, $L<\dfrac{b\left(1-\sqrt{5}\right)}{2}$ and the number of roots is infinite, by Conclusion 3 we get that the value of the expression does not exist.

Problem 2. Determine the value of the expression $\sqrt{4-2\sqrt{4-2\sqrt{4-2\sqrt{\ldots}}}}$, if:

a) the number of the roots is 2017;

b) the number of the roots is 2018.

Solution:

$$b=-2,\ a=4=L(L-b);$$

$$L^2+2L-4=0;$$

$$L_1=-1-\sqrt{5},\ L_2=-1+\sqrt{5}.$$

By Theorem 5 we deduce that $c_{2k-1}=-b=2$ and $c_{2k}=0$.

a) $c_{2017}=c_{2k-1}=2$;

b) $c_{2018}=c_{2k}=0$.

Problem 3. Determine the value of the expression

$$\sqrt{2017-17\sqrt{2017-17\sqrt{2017-17\sqrt{\ldots}}}}.$$

Solution:

$b=-17,\ L(L-b)=L(L+17)=2017;$

$L^2+17L-2017=0;$

$L_{1,2}=\dfrac{-17\pm\sqrt{8357}}{2}.$

From $L>0\Rightarrow L=\dfrac{\sqrt{8357}-17}{2}.$

We compare $\dfrac{\sqrt{8357}-17}{2}$ with $\dfrac{-17\left(1-\sqrt{5}\right)}{2}=\dfrac{17\sqrt{5}-17}{2}=\dfrac{\sqrt{1445}-17}{2}$

$\Rightarrow \dfrac{\sqrt{8357}-17}{2}>\dfrac{\sqrt{1445}-17}{2}.$

From Theorem 2 $\Rightarrow L=\dfrac{\sqrt{8357}-17}{2}$, i.e.

$$\sqrt{2017-17\sqrt{2017-17\sqrt{2017-17\sqrt{\ldots}}}}=\frac{\sqrt{8357}-17}{2}.$$

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ЧИСЛОВИ РЕДИЦИ, СВЪРЗАНИ С КВАДРАТНИ КОРЕНИ (ВТОРА ЧАСТ)

Резюме. Статията е продължение на едноименната публикация в сп. „Математика и информатика“, бр. 5, 2017 г. Теоретичните разглеждания в нея обхващат всички случаи при пресмятане стойностите на краен и безкраен брой вложени квадратни радикали.

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