Educational Technologies
Образователни технологии

SOME NUMERICAL SEQUENCES CONCERNING SQUARE ROOTS (PART ONE)

Rosen Nikolaev, Tanka Milkova, Yordan Petkov
University of Economics – Varna

Abstract. The article is dedicated to one type of problems with an infinite number of nested square radicals proposed in mathematical competitions. A generalization is considered and the deduced theoretical relation is proved by the method of the complete mathematical induction.

Keywords: infinite sequence; nested radical; convergence; mathematical induction.

In the last few years students from our country took an active part in a mathematical competition for the Asian countries – Singapore and Asian Schools Math Olympiad (SASMO). The format of the competition (for students from 8-th to 12-th grade) consists of 25 problems – 15 "multiple choice" problems and 10 "open (numerical) answer" problems. With these regulations it is of crucial importance for the students to have knowledge and skills in solving non-standard problems quickly and correctly. However, it is an indisputable fact that the knowledge in mathematics should be built on logical and theoretical base that prove the correctness of a given statement.

The reason for the appearance of the resent paper is a problem from the SASMO competition which may be summarized and given as a theoretical rule proved by appropriate mathematical knowledge and skills.

Problem 1 (*SASMO 2015, Secondary 2, 8-th grade*). Determine the value of the expression $\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$.

Solution. Let $\sqrt{2+\sqrt{2+\sqrt{2+\dots}}} = L$, $L>0$. Using this substitution and having the squares of both parts of the equation, we get a quadratic equation in terms of L:

$\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}} = L \Big|^2;$

$2 + L = L^2;$

$L^2 - L - 2 = 0;$

$L_1 = -1 \quad L_2 = 2.$

Having L>0, the value of the expression is 2, i.e. $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}} = 2.$

Note. In the proposed solution it is taken for granted that the value of the expression $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$ exists and that it is a finite number. The statements in the following paragraphs will prove the correctness of this assumption.

Similar expressions, known as "nested radicals", have been studied by different authors (Herschfeld, 1935) (Berndt, 1994), (Ramanujan, 2000), (Knyazev, 2016). In a series of two articles we will cover various cases by focusing on inferring conditions in which the value of such an expression exists and it is a finite number.

Let us summarize problem 1.

Problem 2. Given $L = \sqrt{a+b\sqrt{a+b\sqrt{\ldots}}}$. Find a relation between the natural numbers *a* and *b*, such that *L* is a finite natural number.

Solution. Let us take the numerical sequence:

$$c_1 = \sqrt{a},\ c_2 = \sqrt{a+b\sqrt{a}},\ c_3 = \sqrt{a+b\sqrt{a+b\sqrt{a}}},\ \ldots\ c_n = \sqrt{a+b\sqrt{a+b\sqrt{a+\ldots}}},$$

where c_n has exactly n radicals. It is obvious that $L = \lim_{n\to\infty} c_n$, if the limit exists.

First, let us assume that L exists (in the following paragraphs we will prove, that if some conditions, stated there, are satisfied then *L* exists). If so, proceeding from:

$$(*)\ c_n = \sqrt{a+bc_{n-1}}$$

$$(**)\ \lim_{n\to\infty} c_n = \lim_{n\to\infty} c_{n-1} = L,$$

after a limit procedure, we can conclude that:

$$L = \sqrt{a+bL}. \qquad \textbf{(1)}$$

Calculating the squares of both parts of this equation, we obtain a quadratic equation in terms of *L*:

$$L = \sqrt{a + bL} \,\Big|^2 \quad \Leftrightarrow \quad L^2 - bL - a = 0;$$

$$L_{1,2} = \frac{b \pm \sqrt{b^2 + 4a}}{2}.$$

Initially it is given that $L \in \mathbb{N}$, hence $L = \frac{b + \sqrt{b^2 + 4a}}{2}$ $\left(\text{obviously } \frac{b - \sqrt{b^2 + 4a}}{2} < 0\right)$ or $\sqrt{b^2 + 4a} = 2L - b$. For $L \geq \frac{b}{2}$ we raise both sides to the second power:

$$b^2 + 4a = 4L^2 - 4Lb + b^2$$

and we can conclude the following:

$$a = L(L - b), L > b. \quad \textbf{(2)}$$

This relation shows, that for each a there exist (and quite often more than one) numbers L and b, which satisfy**(2)**. Really:

– If $a = p$ – prime number, then $L = p$ and $b = p - 1$.

– If a is not a prime number and d_1, d_2 are its factors ($d_1 > d_2$), then for example $L = d_1$ и $b = d_1 - d_2$.

From another point of view we can always construct a problem from that type regardless of the value of L, given in advance. Hence, whatever natural number $b < L$ is chosen, $a = L(L - b)$.

This makes possible formulation of different problems for mathematical competitions.

Now let us show that in terms of **(2)** the solution of the problem is a natural number.

Let us take for example the following problem:

Problem 3. Determine the value of the expression

$$\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{t(t-b) + \ldots}}}, \quad \textbf{(3)}$$

where t and b are natural numbers ($t > b$).

Solution. Let us assume once again, that this value exists and it is equal to $L\,(L < \infty)$. We reach **(1)** in the form $L = \sqrt{t(t-b) + bL}$ or $L^2 - bL - t(t-b) = 0$, and the discriminant of the last equation is $D = b^2 + 4t^2 - 4tb = (b - 2t)^2$. Then $L_{1,2} = b \pm (b - 2t)$, $L_1 = b - t$, $L_2 = t$ and from $b - t < 0$, we have $L = t \in \mathbb{N}$.

We still have to show that the sequence $\{c_n\}_{n=1}^{\infty}$:

$$c_1 = \sqrt{t(t-b)}, \quad c_2 = \sqrt{t(t-b) + b\sqrt{t(t-b)}}, \ldots, \quad c_n = \underbrace{\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{\ldots}}}}_{n \text{ radicals}}$$

is convergent in terms of the conditions:

(*) $t, b \in \mathbb{N}$;

(**) $t > b$

and its limit is equal to *t*.

Firstly, we will show that the sequence has an upper limit *t*. We will use a method that authors often choose to prove inequalities (see e.g. Miryanov & Petkov, 2017), known as "complete mathematical induction".

1. We check the correctness of the inequality $c_1 < t$;

$c_1 = \sqrt{t(t-b)} < t$ **(4)** $\Leftrightarrow \quad t^2 - bt < t^2 \quad \Leftrightarrow \quad bt > 0$ – correct, because $t, b \in \mathbb{N}$;

A similar equality is correct for the second element of the sequence:

$$c_2 = \sqrt{t(t-b) + b\sqrt{t(t-b)}} < t \quad \Leftrightarrow \quad t^2 - tb + b\sqrt{t^2 - tb} < t^2 \quad \Leftrightarrow \quad \sqrt{t^2 - tb} < t$$

– correct, concluded from **(4)**.

2. Let us assume that the inequality $c_k < t$ is correct for each $k = 1,2,...n$, i.e.

$$\underbrace{\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{\ldots}}}}_{k \text{ radicals}} < t \text{ for } k = 1,2,...n.$$

3. We will prove that the inequality $c_{n-1} < t$ is correct too. We have:

$$c_{n+1} = \underbrace{\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{\ldots}}}}_{n+1 \text{ radicals}} = \sqrt{t(t-b) + b\underbrace{\sqrt{t(t-b) + b\sqrt{\ldots}}}_{n \text{ radicals}}} < \sqrt{t(t-b) + bt} = t$$

- correct.

Hence $c_n < t$ **(5)** for each natural number *n*. This proves that the sequence c_n has an upper limit *t*.

Secondly, we will show that c_n is a monotonic increasing sequence, i.e. $c_n > c_{n-1}$ for each $n = 1,2,...$:

$$c_n = \sqrt{t(t-b) + bc_{n-1}} > c_{n-1} \textbf{ (6)} \Leftrightarrow \quad c_{n-1}{}^2 - bc_{n-1} - t(t-b) < 0 .$$

The solution of the last inequality is $c_{n-1} \in (k_1, k_2)$, where k_1 and k_2 are the roots of the equation $c_{n-1}^2 - bc_{n-1} - t(t-b) = 0$ in terms of the variable c_{n-1}.

$$k_{1,2} = \frac{b \pm \sqrt{b^2 + 4t - 4bt}}{2} = \frac{b \pm (b - 2t)}{2};$$

$k_1 = t > 0$ and $k_2 = b - t < 0$.

Hence $c_{n-1} \in (b - t, t)$.

From that we can make the conclusion that the inequality **(6)** is correct if $c_{n-1} \in (b-t,t)$ but this is a conclusion from **(5)**, where we proved that $c_n < t$ for every natural *n*.

So we proved that $\{c_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence which is bounded above. From the Theorem for convergence of monotonic bounded sequence (Ilin, Sadovnichi & Sendov, 1984), we can conclude that it is convergent. From another point of view it was proved that if its upper limit is *L*, then $L = t$, which solves the problem.

Note. It can be seen from the written above that if the conditions $t,b \in \mathbb{N}$, are not present, then for the convergence of $\{c_n\}_{n=1}^{\infty}$, it will be sufficient if $t > b > 0$. In this case its limit is again *t*. This is a new summary of the initial problem.

Based on all used in the proof of Problem 3, the following Theorem can be formulated:

Theorem 1. If t and b are positive real numbers and $t > b$, then the value of the expression $\sqrt{t(t-b)+b\sqrt{t(t-b)+b\sqrt{\ldots}}}$ is equal to *t*.

The following problems can be easily solved using Theorem 1:

Problem 4. Determine the value of the expression $\sqrt{2018+1007\sqrt{2018+1007\sqrt{\ldots}}}$

Solution. $b = 1007;\ 2018 = t(t-1007) = 1009.2 \quad \Rightarrow \quad t = 1009.$ From Theorem 1 it can be concluded, that the value of the expression is 1009.

Problem 5. Determine for which natural numbers n the following equality is satisfied

$$\sqrt{n+27\sqrt{n+27\sqrt{n+27\sqrt{\ldots}}}} = 30.$$

Solution. Applying Theorem 1 again, we determine:

$$t = 30, b = 27 \Rightarrow t - b = 3 \Rightarrow n = 90.$$

We leave the following problems to the reader.

Problem 6 (*SASMO 2015, Secondary 3, 9-th grade*).

Determine the value of the expression $\sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$.

Problem 7 (*SASMO 2015, Secondary 4, 10 - 12 grade*).

Determine the value of the expression $\sqrt{12+\sqrt{12+\sqrt{12+\ldots}}}$.

Problem 8 (*SASMO 2013, Secondary 3, 9-th grade*).

Determine the value of the expression $\sqrt{2013+28\sqrt{2013+28\sqrt{2013+28\sqrt{\ldots}}}}$.

Problem 9 (*SASMO 2014, Round 1, Secondary*).

Solve the equation $\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}=4.$

NOTES

1. SASMO official website: https://sasmo.sg (accessed August 2017)

SASMO problems:

http://www.apta-academy.org/2013/06/download-international-olympiad/ (accessed August 2017);

https://www.intereseducation.com/resource-tags/sasmo-answers-and-solutions/ (accessed August 2017).

2. Knyazev, A. (2016). Beskonechnye vlozhennye korni. Matematicheskie melochi. http://4ege.ru/matematika/53131-beskonechnye-vlozhennye-korni.html (accessed August 2017). (In Russian).

REFERENCES

Berndt, B. C. (1994). *Ramanujan's Notebooks*, Part IV. New York: Springer-Verlag, p. 14 – 20.

Herschfeld, A. (1935). On Infinite Radicals. *Amer. Math. Monthly*, 42, 419 – 429.

Ilin, V., V. Sadovnichi & B. Sendov (1984). *Matematicheski analiz – parva chast*, Sofia: Nauka i izkustvo. (In Bulgarian).

Grozdev, S. (2007). *For High Achievements in Mathematics.The Bulgarian Experience (Theory and Practice)*. Sofia: ADE, 295 pages (ISBN 978-954-92139-1-1).

Grozdev, S., Ts. Bajcheva, P. Piperkov K. Kirilova-Lupanova (2008). *School-leaving examination in Mathematics*, V. Tirnovo: Abagar, 108 pages (ISBN 978-954-427-782-6). (In Bulgarian).

Grozdev, S. Ts. Bajcheva (2016). *University entering exams in Mathematics*. Sofia: Mathematics Plus, 72 pages (ISSN 0861-8321). (In Bulgarian).

Miryanov, R. & J. Petkov (2017). An application of the relation between arithmetic mean and geometric mean for rational proving some inequalities. *Mathematics and Informatics*, 4 (60), 363 – 369. (In Bulgarian).

Ramanujan, S. (2000). *Collected Papers of Srinivasa Ramanujan* (Ed. G. H. Hardy, P. V. S. Aiyar, and B. M. Wilson). Providence, RI: Amer. Math. Soc., p. 327.

ЧИСЛОВИ РЕДИЦИ, СВЪРЗАНИ С КВАДРАТНИ КОРЕНИ (ПЪРВА ЧАСТ)

Резюме. Статията е посветена на един клас задачи с безброй много вложени квадратни радикали, предлагани на математически състезания. Разгледано е обобщение и изведената теоретична зависимост е доказана с метода на пълната математическа индукция.

[1] Dr. Rosen Nikolaev, Assoc. Prof.,
[2] Dr. Tanka Milkova, Assoc. Prof.,
[3] Dr. Jordan Petkov, Assist. Prof.
University of Economics – Varna
77, Kniaz Boris I Blvd.
9002 Varna, Bulgaria
E-mail: [1] nikolaev_rosen@ue-varna.bg,
[2] tankamilkova@ue-varna.bg,
[3] jr_petkov@ue-varna.bg