

SOME NEW QUESTIONS ABOUT SEQUENCES OF SEMI-PRIME AND PRIME NUMBERS

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Abstract. Everywhere in Nature and life, prime numbers are used very often: cicadas build their life cycles on them, watchmakers use them to calculate ticks, and in aircraft engines with their help the frequency of air pulses is balanced. However, all these areas of application fade against the backdrop of a fact familiar to every cryptographer: prime numbers are in the very heart of modern computer security, that is, they are directly responsible for protecting everything. If you see the lock in the address bar of the browser, it means that a two-key “handshake” is used, based on primes. And our credit cards are also protected using cryptography based on primes.

During many years prime numbers pull the attention of many mathematicians in the world. There is a number of well-known open questions concerning prime and semi-prime numbers. In this paper, some statements are proposed and new open questions are presented concerning sequences of semi-prime and prime numbers.†

Keywords: prime numbers; semi-prime numbers; applications

Introduction

Prime numbers have fascinated all mathematicians and not only them for a long time. They participate in dozens of important applications. Cicadas time their life cycles by prime numbers, modern screens use them to define color intensities of pixels, and manufacturers use primes to get rid of harmonics in their products. However, these uses pale in comparison to the fact that the prime numbers make up the very basis of modern computational security. They are an irreducible part of the very fabric of the Universe.

Using widely prime numbers in cryptography does not happen occasionally. In the most commonly used public-key cryptography system, both the public and the private keys are derived from pairs of large prime numbers according to a relatively simple mathematical formula. In theory, it might be possible to derive the private key from the public key by working the formula backwards. But only the product of large prime numbers is public, and factoring numbers of that size into primes is so hard that even the most powerful supercomputers in the world cannot break an ordinary public key (Schneier, 2015).

Euclid defined a prime number as “that which is measured by the unit only”. The journey from Euclid’s discovery that there are infinitely many primes to the so-

phisticated investigation of prime numbers in the twenty-first century has many important milestones like “The sieve of Eratosthenes”, “The prime counting function of Gauss”, “The smooth analogue of prime number theorem of Chebyshev”, “The zeta function of Riemann” and many others. The mystery of prime numbers pulls attention of many mathematicians from all over the world for many centuries. The formulated by Edmund Landau four problems (1912) challenged several generations of mathematicians to solve them (Grozdev & Nenkov, 2018). The problems are the following and still they are unsolved:

1. Goldbach’s conjecture: Is it possible for every even integer greater than 2 to be presented as the sum of two primes?
2. Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime too?
3. Legendre’s conjecture: Does it always exist one prime at least between consecutive perfect squares?
4. Are there infinitely many primes of the form $n^2 + 1$?

One of the factors that mathematicians were easily encouraged in coping with these problems is: all the problems are understandable even for young students and the problems seem very easy to be solved. This fact creates some mental dissonance between how they look like and the problems’ difficulty. The unenthusiastic fact for all of us is: in last more than one hundred years many mathematicians tried to solve these problems without any success.

In this paper some thoughts, statements and open questions are presented concerning prime and semi-prime numbers.

Semi-prime numbers

In modern parlance a prime number is an integer greater than 1 that is only divisible by itself and 1, such as 2, 3, 5, 7, 11, 13, and so on. Prime numbers are the building blocks of multiplication, the smallest multiplicative units. A number is prime if it cannot be presented as the product of two smaller numbers, and all the composite numbers can be constructed by multiplying a unique set of primes together.

In mathematics, a semi-prime number (also called biprime) is a natural number, which is a product of exactly two (not necessarily distinct) prime numbers. The semi-primes less than 100 are 4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, 34, 35, 38, 39, 46, 49, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, and 95. Semi-primes that are not perfect squares are called discrete, or distinct, semi primes.

Statement 1. The longest sequence of consecutive discrete semi-prime numbers contains 3 members only.

Proof: First, we will show the existence of such a triple: 33 (3×11), 34 (2×17), 35 (5×7). Now, we should prove that such sequences, containing more than 3 members, do not exist. Suppose that such a sequence ($N = 4$) exists. Then, one of the members should be divided by 4 and consequently by two 2-es. Therefore, it will not be a semi prime number. We should take into account the sequences, containing the number

4 itself. Then, we need to check the next sequences: $\{2,3,4,5\}$, $\{3,4,5,6\}$ and $\{4,5,6,7\}$. But all of them do not fit the necessary condition.

There are 14 sequences among the first thousand natural numbers (1 – 1000): $\{33,34,35\}$; $\{85,86,87\}$; $\{93,94,95\}$; $\{121,122,123\}$; $\{141,142,143\}$; $\{201,202,203\}$; $\{213,214,215\}$; $\{217,218,219\}$; $\{301,302,303\}$; $\{393,394,395\}$; $\{445,446,447\}$; $\{633,634,635\}$; $\{841,842,843\}$; $\{921,922,923\}$.

Open Question 1: Does it exist an infinite number of triplets of consecutive semi-prime numbers (like $\{33,34,35\}$; $\{85,86,87\}$; $\{93,94,95\}$ and so on...)?

Prime and semi-prime numbers

Question 1: Does it exist a triplet of *consecutive* natural numbers such that one of them is a prime number (P) and the others two are semi-primes (S)? If so, what is their possible order ($\{P,S,S\}$, $\{S,P,S\}$ or $\{S,S,P\}$)?

In the case $\{S,P,S\}$ there is only one triple $\{4,5,6\}$ which fits the condition. Indeed, because P should not be divisible by 2, then both the semi prime numbers should be even and one of them should be divisible by 4. Hence, the number 4 itself should be included in the sequence. Checking all the triplets of consecutive numbers in the SPS order including 4, leads us to the unique case $\{4,5,6\}$.

Possible triplets in the case $\{P,S,S\}$ are $\{13,14,15\}$; $\{157,158,159\}$ and so on...

Open Question 2: Does it exist an infinite number of triplets of consecutive numbers in order $\{P,S,S\}$ (like $\{13,14,15\}$)?

Possible triplets in the case $\{S,S,P\}$ are $\{9,10,11\}$; $\{21,22,23\}$ and so on...

Open Question 3: Does it exist an infinite number of triplets of consecutive numbers in order $\{S,S,P\}$ (like $\{9,10,11\}$)?

Question 2: Does it exist tripe of *consecutive* natural numbers such that one of them is a semi-prime number and the other two are prime? If so, what is the possible order of the numbers ($\{P,P,S\}$, $\{P,S,P\}$ or $\{S,P,P\}$)?

It is clear that in the case $\{P,P,S\}$ there is only one triplet: $\{2,3,4\}$. It is also clear that there is no triplet in the case of $\{S,P,P\}$. The possible triplets in the case $\{P,S,P\}$ are $\{3,4,5\}$; $\{5,6,7\}$ and so on...

Open Question 4: Does it exist an infinite number of triplets of consecutive numbers in order $\{P,S,P\}$ (like $\{3,4,5\}$)?

Remark. The answer to this question depends on another open question about the existence of twin prime numbers (The twin prime conjecture).

Statement 2. The longest sequence of consecutive natural numbers, that consists of one prime number at least and one semi-prime number at least, has 6 members and it is $\{2,3,4,5,6,7\}$.

Proof: It is clear that in all sequences of 6 *consecutive* numbers, at least one should be divisible by 4 and one should be dividable by 6. Then, it is needed that

the numbers 4 and 6 are present in the sequence themselves (otherwise, the numbers that are divisible by 4 or 6 will not be semi-prime numbers). The unique sequence with such a property is $\{2,3,4,5,6,7\}$.

By analogous reasoning it is clear that it does not exist a sequence of 7 *consecutive* numbers such that, at least one of them is a prime number and at least one of them is a semi-prime number.

Perfect prime numbers

Definition: A perfect prime number is such a prime number that:

- (a) all its digits are prime;
- (b) the sum of its digits is prime.

Open Question 5: Does it exist an infinite number of perfect prime numbers?

Statement 3. There is only one 2-digit perfect prime numbers, namely 23.

Proof: It is clear that such a number cannot contain two odd digits (in this case the sum of the digits is an even number, different from 2, and consequently it is not prime). The first digit should be the digit 2, because other even digits are not prime numbers. Then we should check only 23, 25 and 27 and the answer is: 23.

Remark. *The first 3-digit perfect prime number is 223. Let the reader find other examples. What about 4-digit perfect prime numbers? Answer to the open question 5.*

Conclusion

Some well-known open questions about prime numbers are presented in the paper. Stated many years ago whereas the mathematics community such questions remain still challenging. New questions concerning prime and semi-prime numbers are presented in the paper too hoping to motivate curious mathematicians and other mathematics lovers.

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