Educational Research
Научно–методически статии

SOME INEQUALITIES IN THE TRIANGLE

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Abstract. The paper considers some inequalities in the triangle connecting the semi-perimeter, the circumradius and the inradius.
Keywords: triangle, semi-perimeter, circumradius, inradius, inequality

The following inequalities from (Bottema et al., 1969) (5.5, 5.8, 5.11 and 5.12) are considered in the paper:

(1) $$s^2 \geq 3r(4R+r),$$

(2) $$s^2 \geq r(16R-5r),$$

(3) $$s^2 \geq 27r^2,$$

(4) $$s^2 \geq \frac{27}{2}Rr\cdot$$

The inequalities (1) – (4) are proved in (Bottema et al., 1969), (Grozdev, 2005), (Grozdev, 2007) and we ommit the proofs here. Our goal is to demonstrate that the inequality (2) is the best one with respect to the others. It is valid the follwing chain:

$$s^2 \geq r(16R-5r) \geq \frac{27}{2}Rr \geq 3r(4R+r) \geq 27r^2\cdot$$

We have $r(16R-5r) \geq \frac{27}{2}Rr \Leftrightarrow 2(16R-5r) \geq 27R \Leftrightarrow 5R \geq 10r \Leftrightarrow R \geq 2r$ and the last is the well-known Euler's inequality, whose several proofs one could find in (Arslanagić, 2004), (Arslanagić, 2006), (Arslanagić, 2008) and (Arslanagić, 2009). In a similar way:

$$\frac{27}{2}Rr \geq 3r(4R+r) \Leftrightarrow R \geq 2r \text{ and } 3r(4R+r) \geq 27r^2 \Leftrightarrow R \geq 2r\,.$$

Now, we put the question: "Does any inequality, which is better than the inequality (2), exist or not?" The answer is affirmative; this is the inequality (10) from (Mitrinović et al., 1989), p. 248, for a non-obtuse triangle:

(5) $$s^2 \geq 2R^2 + 8Rr + 3r^2\,.$$

The proof is as follows:

$$2R^2+8Rr+3r^2 \ge r(16R-5r) \Leftrightarrow 2R^2-8Rr+8r^2 \ge 0 \Leftrightarrow R^2-4Rr+4r^2 \ge 0 \Leftrightarrow (R-2r)^2 \ge 0 .$$

The last inequality is obvious.

Note that the equalities in (1), (2), (3), (4) and (5) hold true if and only if $R = 2r$, i.e. for the equilateral triangle. We propose two other proofs of (5).

Proof 1. We will use the next well-known equalities:

(6) $$a^2+b^2+c^2 = 2\left(s^2-r^2-4Rr\right)$$

(7) $$4F = \left(b^2+c^2-a^2\right)tg\alpha ,$$

where F is the area of ΔABC. The equality (7) could be deduced very easily by using the formulae $F=\frac{bc}{2}\sin\alpha$ and $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$. We have $4F=\left(a^2+c^2-b^2\right)tg\beta$ and $4F=\left(a^2+b^2-c^2\right)tg\gamma$. Apply also the well-known facts:

(8) $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma = \frac{2F}{R^2} ,$$

(9) $$\cos\alpha+\cos\beta+\cos\gamma = 1+\frac{r}{R} .$$

We have: $$a^2+b^2+c^2 = \left(b^2+c^2-a^2\right)+\left(c^2+a^2-b^2\right)+\left(a^2+b^2-c^2\right)=$$

$$\overset{(7)}{=} 4F\left(\frac{1}{tg\alpha}+\frac{1}{tg\beta}+\frac{1}{tg\gamma}\right)=4F\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}\right)=8F\left(\frac{\cos^2\alpha}{\sin 2\alpha}+\frac{\cos^2\beta}{\sin 2\beta}+\frac{\cos^2\gamma}{\sin 2\gamma}\right)$$

and by the Cauchy-Bunyakovski-Schwarz inequality (CBS inequality) we get:

$$a^2+b^2+c^2 \ge 8F\cdot\frac{(\cos\alpha+\cos\beta+\cos\gamma)^2}{\sin 2\alpha+\sin 2\beta+\sin 2\gamma} \overset{(8),(9)}{\ge} 8F\cdot\frac{\left(1+\frac{r}{R}\right)^2}{\frac{2F}{R^2}}$$

$$\overset{(6)}{\Leftrightarrow} 2\left(s^2-r^2-4Rr\right) \ge 4(R+r)^2 \Leftrightarrow s^2 \ge 2R^2+8Rr+3r^2,$$

which ends the proof.

Proof 2. Now we will use (9) and (10) together with

(10) $$a\cos\alpha+b\cos\beta+c\cos\gamma=\frac{2F}{R}.$$

By the CBS inequality e get:

$$(a\cos\alpha+b\cos\beta+c\cos\gamma)\left(\frac{\cos\alpha}{a}+\frac{\cos\beta}{b}+\frac{\cos\gamma}{c}\right)\geq(\cos\alpha+\cos\beta+\cos\gamma)^2,$$

and consequently $\frac{2F}{R}\left(\frac{\cos\alpha}{a}+\frac{\cos\beta}{b}+\frac{\cos\gamma}{c}\right)\geq\left(1+\frac{r}{R}\right)^2$. By the sine law $a=2R\sin\alpha,\ b=2R\sin\beta$ and $c=2R\sin\gamma$. It follows that: $\frac{2F}{R}\cdot\frac{1}{2R}(\cot g\alpha+\cot g\beta+\cot g\gamma)\geq\frac{(R+r)^2}{R^2}$, which is equivalent to

(11) $$\cot g\alpha+\cot g\beta+\cot g\gamma\geq\frac{(R+r)^2}{F}.$$

Additionally, we also have:

$$\cot g\alpha+\cot g\beta+\cot g\gamma=\frac{\cos\alpha}{\sin a}+\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}=\frac{\cos\alpha}{\frac{a}{2R}}+\frac{\cos\beta}{\frac{ab}{2R}}+\frac{\cos\gamma}{\frac{c}{2R}}=2R\left(\frac{\cos\alpha}{a}+\frac{\cos\beta}{b}+\frac{\cos\gamma}{c}\right)=$$

$=2R\left(\frac{b^2+c^2-a^2}{2abc}+\frac{a^2+c^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}\right)=\frac{R}{abc}\left(a^2+b^2+c^2\right)$ and from here, because $abc=4RF$, we get:

$$\cot g+c\cot g\beta+\cot g\gamma=\frac{R}{4RF}\cdot 2\left(s^2-r^2-4Rr\right)=\frac{s^2-r(4R+r)}{2F}.$$

Finally, we get: $\frac{s^2-r(4R+r)}{2F}\geq\frac{(R+r)^2}{F}\Leftrightarrow s^2\geq 2R^2+8Rr+3r^2$, which ends the proof.

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