

## SEEMOUS OLYMPIAD FOR UNIVERSITY STUDENTS

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**Abstract.** The paper considers the results of a recent International Mathematical Olympiad for University students. Solutions of the problems with methodological and heuristic details are proposed.

*Keywords:* Olympiad, problem solving, University student, result.

The 6<sup>th</sup> SEEMOUS (South Eastern European Mathematical Olympiad for University Students) took place from March 6 to March 11, 2012 in Blagoevgrad, Bulgaria. Organizers were the MASSEE (Mathematical Society of South Eastern Europe), The Union of Bulgarian Mathematicians and the South-West University „Neofit Rilski” of Blagoevgrad. The event was launched in 2007 by MASSEE – the organization of the following 9 countries: Albania, Bosnia and Herzegovina, Bulgaria, FYROM (Macedonia), Cyprus, Greece, Moldova, Romania and Serbia. Second time Bulgaria hosted the best University students in Mathematics from the Balkans (the first time was in 2010, when the Olympiad took place in Plovdiv). This time 22 teams were presented: Sofia University (with 2 teams), Plovdiv University, South-West University of Blagoevgrad, „St. Cyril and St. Methodius” University of Veliko Tirnovo, American University of Blagoevgrad, University of Economics of Varna, University of Architecture, Civil Engineering and Geodesy of Sofia (with 2 teams), State University of Bucharest, Politechnical University of Bucharest, Technical University of Iasi (Romania), „Al. I. Cuza” University of Iasi (with 2 teams), „Ovidius” University of Constanta (Romania), Technical University of Cluj-Napoca (Romania), University of Athens (with 2 teams), Democritus University of Thrace (Xanthi, Greece), „Sts. Cyril and Methodius” University of Skopje, Middle East Technical University (Turkey) and „Taras Shevchenko” National University of Kiev (Ukraine). The total number of participants was 148, including 97 students. The International Jury proposed the following paper, consisted of 4 problems (each worth 10 points):

**Problem 1.** (Number Theory) Let  $A = (a_{ij})$  be a  $n \times n$  matrix, where  $a_{ij}$  is the non-negative remainder of the division of  $i^j + j^i$  by 3 for  $i, j = 1, 2, \dots, n$ . Find the

greatest  $n$  for which  $\det A \neq 0$ .

(Proposed by Ukraine, author – **Volodymyr Brayman**  
from the Kyiv Taras Shevchenko National University)

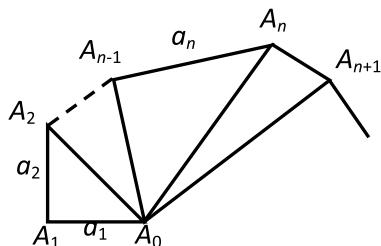
*Solution:* We have  $a_{i+6,j} = a_{ij}$  for all  $i, j = 1, 2, \dots, n$ . To prove this, first note that if  $j \equiv 0 \pmod{3}$ , then  $j^i \equiv 0 \pmod{3}$ . Also, if  $j \equiv 1$  or  $2 \pmod{3}$ , then  $j^6 \equiv 1 \pmod{3}$ . Hence,  $j^i(j^6 - 1) \equiv 0 \pmod{3}$  for  $j = 1, 2, \dots, n$  and

$$a_{i+6,j} \equiv (i+6)^j + j^{i+6} \equiv i^j + j^i \equiv a_{ij} \pmod{3}.$$

Thus,  $a_{i+6,j} = a_{ij}$  and consequently  $\det A = 0$  for  $n \geq 7$ . A straightforward calculation shows that  $\det A = 0$  for  $n = 6$  but  $\det A \neq 0$  for  $n = 5$ . The answer is  $n = 5$ .

**Grading:** **5 p.** for concluding that  $\det A = 0$  for  $n \geq 7$ ; **5 p.** for computing that  $\det A = 12$  for  $n = 5$  and  $\det A = 0$  for  $n = 6$ ; **2p.** (in case none of the previous is done) for computing that  $\det A = -10$  for  $n = 3$  and  $\det A = 4$  for  $n = 4$ .

**Problem 2.** (Geometry) Consider the right triangles:  $\Delta A_0 A_1 A_2, \Delta A_0 A_2 A_3, \dots, \Delta A_0 A_{n-1} A_n, \dots, (n \geq 1)$  with  $|A_{i-1} A_i| = a_i > 0$  for  $i \geq 1$  (see the figure). (More precisely, for each  $i \geq 2$  the hypotenuse  $A_0 A_i$  of  $\Delta A_0 A_{i-1} A_i$  is a leg of  $\Delta A_0 A_i A_{i+1}$  with right angle  $\angle A_0 A_i A_{i+1}$  and the vertices  $A_{i-1}$  and  $A_{i+1}$  lie in different semi-planes with respect to the straight line



$A_0 A_i$ .) Is it possible for the set of points  $\{A_n : n \geq 0\}$  to be unbounded but the series  $\sum_{n=2}^{\infty} m(\angle A_{n-1} A_0 A_n)$  to be convergent, where  $m(\angle ABC)$  denotes the measure of  $\angle ABC$ ?

*Note.* A subset  $B$  of the plane is bounded if there is a disk  $D$  such that  $B \subseteq D$ .

(Proposed by Ukraine, author – **Volodymyr Brayman**  
from the Kyiv Taras Shevchenko National University)

*Solution:* We have  $|A_0 A_n| = \sqrt{\sum_{i=1}^n a_i^2}$  and  $\sum_{n=2}^k m(\angle A_{n-1} A_0 A_n) = \sum_{n=2}^k \arctan \frac{a_n}{\sqrt{a_1^2 + \dots + a_{n-1}^2}}$

The set of points  $\{A_n : n \geq 0\}$  will be unbounded iff the sequence of the lengths of the

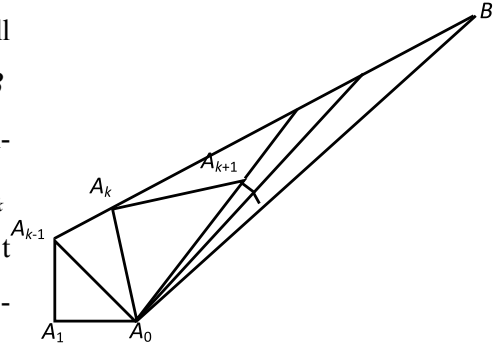
segments  $A_0A_n$  is unbounded. Put  $a_i^2 = b_i$ . Then, the question of the problem could be reformulated as follows: Is it possible for a series with positive terms to be such that

$$\sum_{i=1}^{\infty} b_i = \infty \text{ and } \sum_{n=2}^{\infty} \arctan \sqrt{\frac{b_n}{b_1 + \dots + b_{n-1}}} < \infty.$$

Denote  $s_n = \sum_{i=1}^n b_i$ . Since  $\arctan x \sim x$  as  $x \rightarrow 0$ , the question we need to ask is whether one can have  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\sum_{n=2}^{\infty} \sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} < \infty$ . Let  $\sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} = u_n > 0$ . Then  $\frac{s_n}{s_{n-1}} = 1 + u_n^2 \Rightarrow \ln s_n - \ln s_{n-1} = \ln(1 + u_n^2) \Rightarrow \ln s_k = \ln s_1 + \sum_{n=2}^k \ln(1 + u_n^2)$ . Finally, the question is whether it is possible to have  $\sum_{n=2}^{\infty} \ln(1 + u_n^2) = \infty$  and  $\sum_{n=2}^{\infty} u_n < \infty$ . The answer is negative, since  $\ln(1 + x) \sim x$  as  $x \rightarrow 0$  and  $u_n^2 \leq u_n \leq 1$  for large enough  $n$ .

*Alternative solution:* Since  $\sum_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n) < \infty$ , there exists some large enough  $k$

for which  $\sum_{n=k}^{\infty} m(\angle A_{n-1}A_0A_n) \leq \beta < \frac{\pi}{2}$ . Then all the vertices  $A_n$  ( $n \geq k-1$ ) lie inside  $\triangle A_0A_{k-1}B$ , where the side  $A_{k-1}B$  of  $\triangle A_0A_{k-1}B$  is a continuation of the side  $A_{k-1}A_k$  of  $\triangle A_0A_{k-1}A_k$  and  $\angle A_{k-1}A_0B = \beta$ . Consequently, the set  $\{A_n : n \geq 0\}$  is bounded, which is a contradiction.



**Grading:** **1 p.** for noting that  $\{A_n : n \geq 0\}$  is unbounded  $\Leftrightarrow |A_0A_n|$  is unbounded **OR** for expressing  $|A_0A_n|$ ; **1 p.** for observing that  $\sum_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n)$  is convergent  $\Leftrightarrow A_0A_n$  tends to  $A_0B$  **OR** for expressing the angles by  $\arctan g$ ; **8 p.** for proving the assertion.

**Problem 3.** (Algebra) a) Prove that if  $k$  is an even positive integer and  $A$  is a real symmetric  $n \times n$  matrix such that  $(\text{Tr}(A^k))^{k+1} = (\text{Tr}(A^{k+1}))^k$ , then  $A^n = \text{Tr}(A)A^{n-1}$ .

b) Does the assertion from a) hold true for odd positive integers  $k$ ?

(Proposed by Romania, author – **Vasile Pop**  
from the Technical University of Cluj-Napoca)

*Solution:* a) Let  $k = 2l$  ( $l \geq 1$ ). Since  $A$  is symmetric, all its eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  are real numbers. We have:

$$(1) \quad \text{Tr}(A^{2l}) = \lambda_1^{2l} + \lambda_2^{2l} + \dots + \lambda_n^{2l} = a$$

and

$$(2) \quad \text{Tr}(A^{2l+1}) = \lambda_1^{2l+1} + \lambda_2^{2l+1} + \dots + \lambda_n^{2l+1} = b.$$

By (1) we get that  $a \geq 0$ , so there is some  $a_1$  such that  $a = a_1^{2l}$ . On the other hand, the equality  $a^{2l+1} = b^{2l}$  implies that  $(a_1^{2l+1})^{2l} = b^{2l}$  and hence:

$$b = \pm a_1^{2l+1} = (\pm a_1)^{2l+1} \quad \text{and} \quad a = a_1^{2l} = (\pm a_1)^{2l}.$$

Thus, equalities (1) and (2) become:

$$(3) \quad \lambda_1^{2l} + \lambda_2^{2l} + \dots + \lambda_n^{2l} = c^{2l}$$

and

$$(4) \quad \lambda_1^{2l+1} + \lambda_2^{2l+1} + \dots + \lambda_n^{2l+1} = c^{2l+1},$$

where  $c = \pm a_1$ . We consider the following cases:

Case 1. If  $c = 0$ , then  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ , so  $\text{Tr}(A) = 0$  and we note that the characteristic polynomial of  $A$  is  $f_A(x) = x^n$ . Applying the Cayley-Hamilton theorem (Lang, 1965) we get  $A^n = 0 = \text{Tr}(A)A^{n-1}$ .

Case 2. If  $c \neq 0$ , then let  $x_i = \frac{\lambda_i}{c}$  ( $i = 1, 2, \dots, n$ ). In this case (3) and (4) become:

$$(5) \quad x_1^{2l} + x_2^{2l} + \dots + x_n^{2l} = 1$$

and

$$(6) \quad x_1^{2l+1} + x_2^{2l+1} + \dots + x_n^{2l+1} = 1.$$

The equality (5) implies that  $|x_i| \leq 1$  for all  $i = 1, 2, \dots, n$ . We have  $x^{2l} \geq x^{2l+1}$  for  $|x| \leq 1$  with equality reached when  $x = 0$  or  $x = 1$ . Then, by (5), (6) and the previous observation we find without loss of generality that  $x_1 = 1, x_2 = x_3 = \dots x_n = 0$ . Hence  $\lambda_1 = c, \lambda_2 = \lambda_3 = \dots = \lambda_n = 0$ . This implies that  $f_A(x) = x^{n-1}(x - c)$  and  $Tr(A) = c$ . Again, it follows by the Cayley-Hamilton theorem that

$$f_A(A) = A^{n-1}(A - cI_n) = 0 \Leftrightarrow A^n = Tr(A)A^{n-1}.$$

b) The answer is negative. It is enough to consider the following example:

$$k = 1, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

**Grading:** 3 p. for a reformulation of the problem using eigenvalues:

$$\left(\sum \lambda_i^{2l}\right)^{2l+1} = \left(\sum \lambda_i^{2l+1}\right)^{2l} \Rightarrow \forall i: \lambda_i^n = (\lambda_1 + \lambda_2 + \dots + \lambda_n) \lambda_i^{n-1};$$

4 p. for proving that only  $(\lambda_i) = (0, \dots, 0, c, 0, \dots, 0)$  or  $(0, \dots, 0)$  are possible; 3

p. for finding a counter example.

**Problem 4.** (Analysis) a) Compute  $\lim_{n \rightarrow \infty} n \int_0^1 \left( \frac{1-x}{1+x} \right)^n dx$ .

b) If  $k \geq 1$  is integer, compute  $\lim_{n \rightarrow \infty} n^{k+1} \int_0^1 \left( \frac{1-x}{1+x} \right)^n x^k dx$ .

(Proposed by Romania, author – **Ovidiu Furdui**  
from the Technical University of Cluj-Napoca)

*Solution:*

a) The answer is  $\frac{1}{2}$ . It follows immediately from b) when  $k = 0$ .

b) The answer is  $\frac{k!}{2^{k+1}}$ . By the substitution  $\frac{1-x}{1+x} = y$  we get:

$$n^{k+1} \int_0^1 \left( \frac{1-x}{1+x} \right)^n x^k dx = 2n^{k+1} \int_0^1 y^n (1-y)^k \frac{dy}{(1+y)^{k+2}} = 2n^{k+1} \int_0^1 y^n f(y) dy, \text{ where } f(y) = \frac{(1-y)^k}{(1+y)^{k+2}}.$$

Observe that:

$$(7) \quad f(1) = f'(1) = \dots = f^{(k-1)}(1) = 0.$$

Integrate  $\int_0^1 y^n f(y) dy$  by parts  $k$  times. Then, by (7) we get:

$$\int_0^1 y^n f(y) dy = \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)} \int_0^1 y^{n+k} f^{(k)}(y) dy.$$

One more integration implies that:

$$\int_0^1 y^n f(y) dy = \frac{(-1)^k f^{(k)}(1)}{(n+1)(n+2)\dots(n+k+1)} \int_0^1 y^{n+k+1} f^{(k+1)}(y) dy.$$

It follows that:

$$\lim_{n \rightarrow \infty} 2n^{k+1} \int_0^1 y^n f(y) dy = 2(-1)^k f^{(k)}(1), \text{ since } \lim_{n \rightarrow \infty} \int_0^1 y^{n+k+1} f^{(k+1)}(y) dy = 0,$$

$f^{(k+1)}(y)$  being continuous and hence bounded. Using Leibniz' s formula, we get

that  $f^{(k)}(1) = (-1)^k \frac{k!}{2^{k+2}}$ , thus solving the problem.

**Grading:** 3 p. for computing a); 7 p. for computing b).

The International Jury distributed 66 medals: 12 gold, 21 silver and 33 bronze. The winner was Victor Padureanu from the Politechnical University of Bucharest who received 34 points out of 40. A special prize was awarded to the silver medallist Konstantinos Tsouvalas from the University of Athens for his original solution of Problem 4. Applying the Absolute convergence theorem the prize winner made a limit transition under the integral, which allowed him to use properties of the Gamma function. Such an approach demonstrated a high scientific background. Only two Bulgarian students received gold medals, namely Mladen Valkov and Yavor Papazov from the Sofia University, who were 9<sup>th</sup> and 11<sup>th</sup> in the official ranking, respectively. Both of them are well-known participants in Mathematical Olympiads since their school years. Unfortunately, the best Bulgarian secondary school students were not present at SEEMOUS'2012 because of realizing their University studies in Western Europe and USA.

## References

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## Олимпиадата SEEMOUS за студенти

**Резюме.** Статията разглежда резултатите от неотдавнашна международна студентска олимпиада по математика. Предложени са решения на задачите от олимпиадата с методически и евристични детайли.

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