2.1) If $ABCD$ is a trapezoid with $AB \parallel CD$ and $BC \cap AD = W$, then the common point of the diagonal AC and the circumcircle $k(BDW)$ of ΔBDW is the desired point X. The proof is the following:

i) $\angle XAB = \angle XCD$ (cross-opposite angles);

ii) $\angle XBC = \dfrac{\overset{\frown}{XDW}}{2} = \dfrac{\overset{\frown}{XD} + \overset{\frown}{DW}}{2} = \angle XDA$.

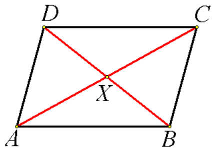
Reversely, the equality $\angle XAB = \angle XCD$ implies, that the point X belongs to the diagonal AC, while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of the line AC and $k(BDW)$ is such, that $\angle YAB = 180^\circ - \angle YCD$ and $\angle YBC = 180^\circ - \angle YDA$.

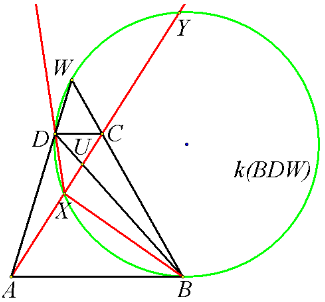
2.2) If $ABCD$ is a trapezoid with $BC \parallel AD$ and $AB \cap CD = V$, then the common point of the diagonal BD and the circumcircle $k(ACV)$ of ΔACV is the desired point X. The proof is the following:

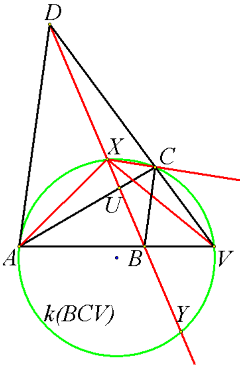
i) $\angle XAB = \dfrac{\overset{\frown}{XCV}}{2} = \dfrac{\overset{\frown}{XC} + \overset{\frown}{CV}}{2} = \angle XCD$;

ii) $\angle XBC = \angle XDA$ (cross-opposite angles).

The uniqueness of the point X could be proved in the same manner as in the previous case. The second common point Y of the line BD and $k(ACV)$ is such, that $\angle YAB = 180^\circ - \angle YCD$ and $\angle YBC = 180^\circ - \angle YDA$.



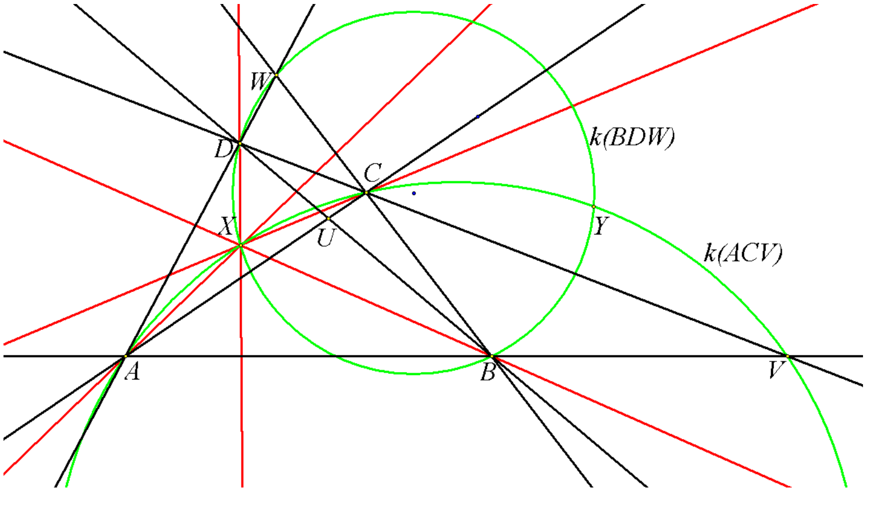




3) If $ABCD$ is without parallel sides and $AB \cap CD = V$, $BC \cap AD = W$, then the interior point of $ABCD$, which is the intersection point of the circumcircles $k(ACV)$ and $k(BDW)$ of ΔACV and ΔBDW, respectively, is the desired point X. The proof is the following:

$$\text{i) } \angle XAB = \frac{\overset{\frown}{XCV}}{2} = \frac{\overset{\frown}{XC} + \overset{\frown}{CV}}{2} = \angle XCD;$$

$$\text{ii) } \angle XBC = \frac{\overset{\frown}{XDW}}{2} = \frac{\overset{\frown}{XD} + \overset{\frown}{DW}}{2} = \angle XDA.$$



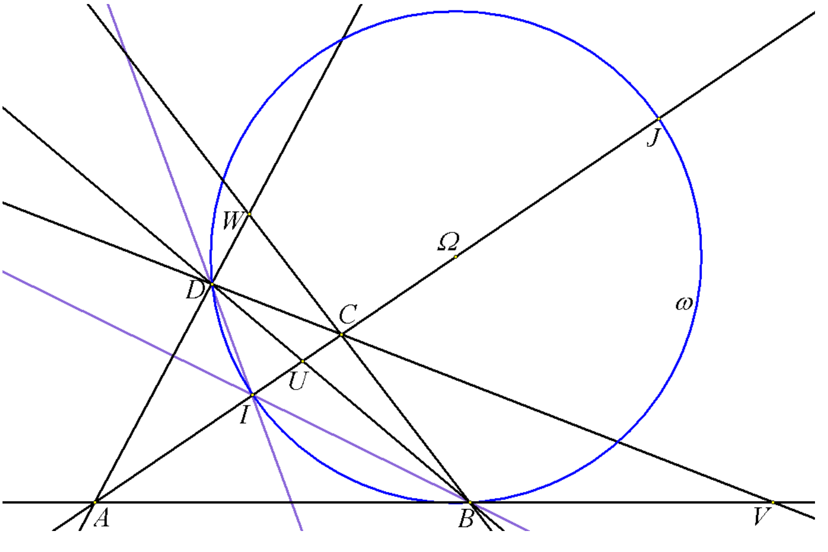
Reversely, the equality $\angle XAB = \angle XCD$ means, that the point X belongs to the circle $k(ACV)$, while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of $k(ACV)$ and $k(BDW)$ is such, that $\angle YAB = 180° - \angle YCD$ and $\angle YBC = 180° - \angle YDA$.

Lemma 2. If the sides of the convex quadrilateral $ABCD$ satisfy the equality $AB.CD = BC.DA$, then:

a) the angular bisectors of the angles ABC and CDA meet in the point I from the diagonal AC;

b) the angular bisectors of the angles BAD and DCB meet in the point I' from the diagonal BD.

Proof: Rewrite the given equality in the form $\frac{BA}{BC} = \frac{DA}{DC}$. If I is such a point on the diagonal AC, that $\frac{IA}{IC} = \frac{BA}{BC}$, then BI is the angular bisector of $\angle ABC$. On the other hand, the equality $\frac{IA}{IC} = \frac{DA}{DC}$ is true, which means, that DI is the angular bisector of $\angle CDA$. The assertion b) could be obtained analogously.

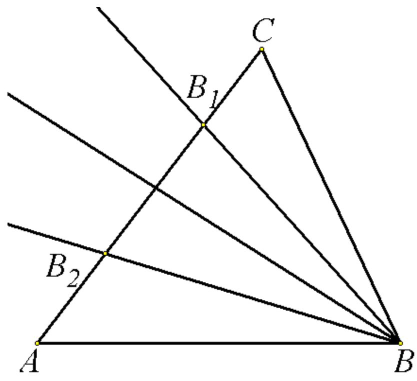


In fact, if $BA \neq BC$, then there exists a point J on the line BC with the property $\frac{JA}{JC} = \frac{BA}{BC}$. The circle ω with diameter IJ is the locus of the points M, for which $\frac{MA}{MC} = \frac{BA}{BC}$. The circle ω is known to be the *Apollonius circle* for the segment AC under the ratio $\frac{BA}{BC}$. For this reason, if an arbitrary ΔABC is given, then the point D from the Apollonius circle ω is the fourth vertex of the quadrilateral $ABCD$, for which $AB.CD = BC.DA$. In such a way we come upon an idea for the construction of a quadrilateral satisfying the conditions of the problem under consideration. On the other hand, lemma 1 gives a possibility to construct the point X for this quadrilateral.

Lemma 3. Let ABC be an arbitrary triangle, while B_1 and B_2 be such points on the line AC, that the lines BB_1 and BB_2 are symmetric with respect to the angular bisector of $\angle ABC$. Then $\frac{\overline{CB_1}}{\overline{AB_1}}.\frac{\overline{CB_2}}{\overline{AB_2}} = \frac{BC^2}{AB^2}$.

Proof: For the areas of the triangles BCB_1 and BAB_2 we have, that $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{CB_1}{AB_2}$ and $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{BC.BB_1.\sin\angle B_1BC}{AB.BB_2.\sin\angle B_2BA} = \frac{BC.BB_1}{AB.BB_2}$. Consequently $\frac{CB_1}{AB_2} = \frac{BC.BB_1}{AB.BB_2}$.

Analogously, using the areas of the triangles CBB_2 and ABB_1 we get the equality $\frac{CB_2}{AB_1} = \frac{BC.BB_2}{AB.BB_1}$. Multiplying the obtained equalities we find, that $\frac{CB_1}{AB_1}.\frac{CB_2}{AB_2} = \frac{BC^2}{AB^2}$.

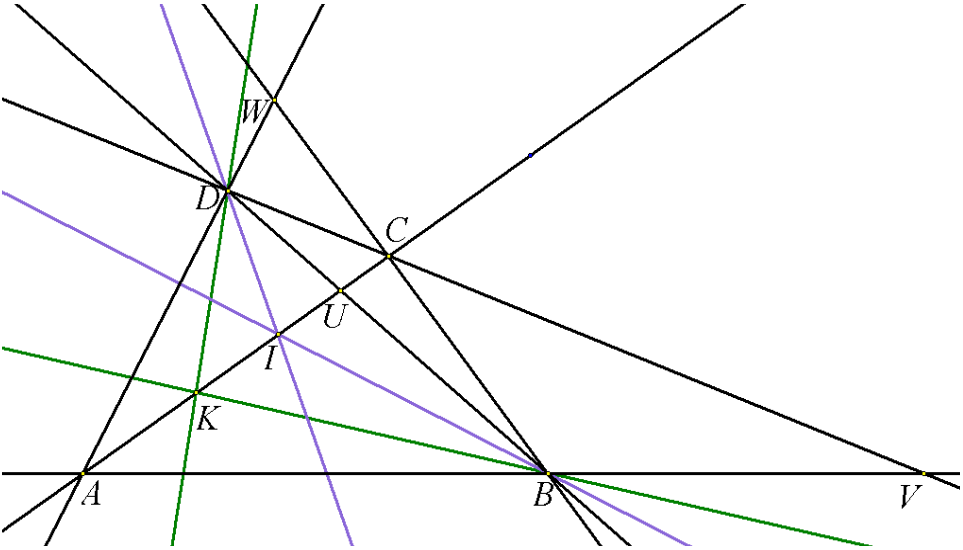


Lemma 4. If the convex quadrilateral $ABCD$ satisfies the equality $AB.CD = BC.DA$, then:

a) the symmetric images of the diagonal BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet in a point K from the diagonal AC;

b) the symmetric images of the diagonal AC with respect to the angular bisectors of $\angle BAD$ and $\angle DCB$ meet in a point K' from the diagonal BD.

Proof: Denote by U the common point of the diagonals AC and BD. Let the symmetric images of BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet AC in the points K_1 and K_2, respectively. Applying lemma 3 to ΔABC we obtain the equality $\frac{CU}{AU}.\frac{CK_1}{AK_1} = \frac{BC^2}{AB^2}$. Applying lemma 3 to ΔCDA we obtain $\frac{AU}{CU}.\frac{AK_2}{CK_2} = \frac{DA^2}{CD^2}$. Multiplying the two equalities we deduce, that $\frac{CK_1.AK_2}{AK_1.CK_2} = \left(\frac{BC.DA}{AB.CD}\right)^2 = 1$. Therefore, $\frac{AK_2}{CK_2} = \frac{AK_1}{CK_1}$. Since the points K_1 and K_2 belong to the segment AC, the last equality means, that the points K_1 and K_2 coincide. The assertion for the point K' could be proven analogously.



It follows from lemma 4 that the point I is the center of the incircle of ΔBDK. Thus, AC is the angular bisector of $\angle BKD$. Consequently $180^\circ = \angle AKD + \angle DKC = \angle AKD + \angle BKC$. Thus we obtain the following

Conclusion. The equalities $\angle AKD + \angle BKC = 180^\circ$ and $\angle BK'A + \angle CK'D = 180^\circ$ are satisfied.

Lemma 5. If the convex quadrilateral $ABCD$ satisfies the equality $AB.CD = BC.DA$, then:

a) The second common point X of the circumcircles $k(BCK)$ and $k(DAK)$ of ΔBCK and ΔDAK, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

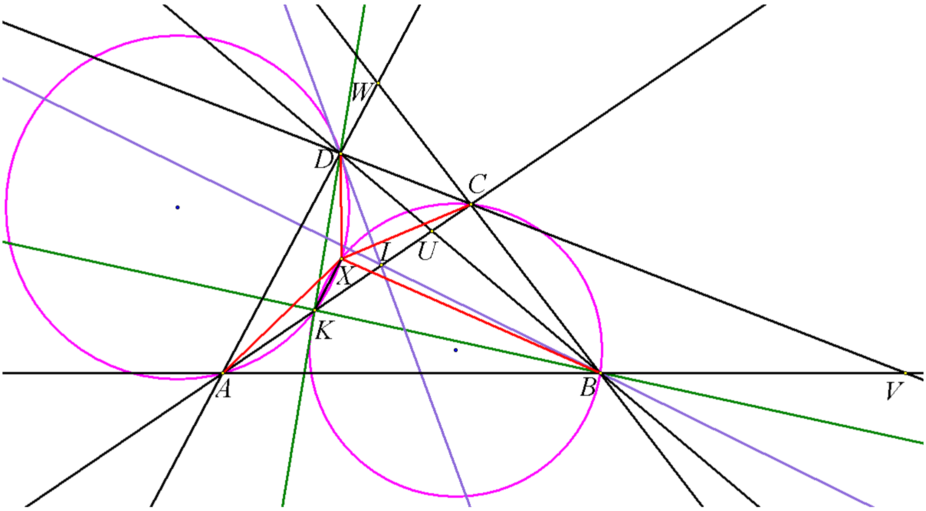
b) The second common point X of the circumcircles $k(BCK')$ and $k(DAK')$ of $\Delta BCK'$ and $\Delta DAK'$, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

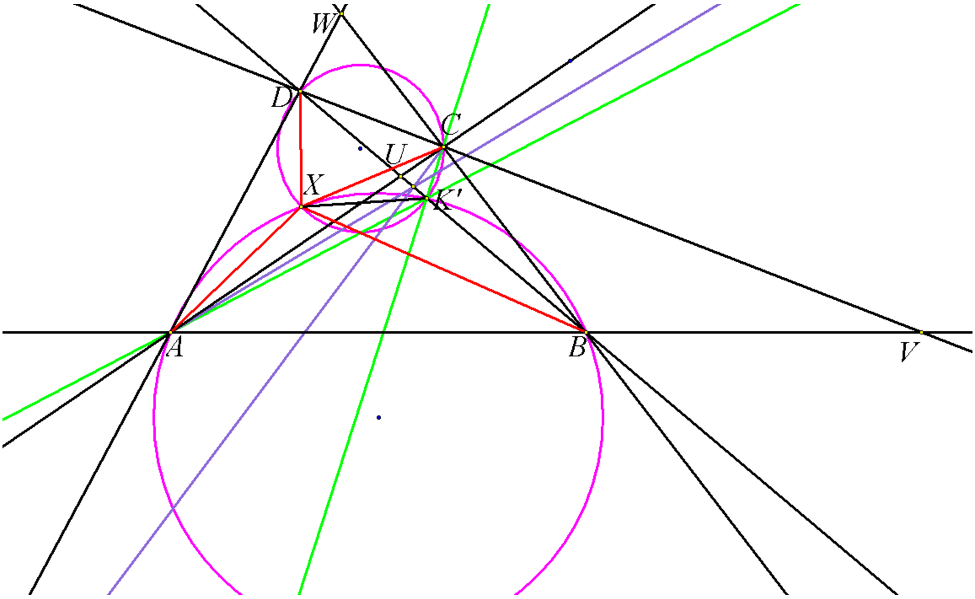
Proof: We will consider the first case only. Let $AB \cap AD = V$ and the second common point of $k(BCK)$ and $k(DAK)$ be X. In order to prove the first equality it is enough to establish, that the point X belongs to the circumcircle $k(ACV)$ of ΔACV (as shown in lemma 1). The following equalities are true:

$$\angle CXA = \angle CXK + \angle KXA = 180^\circ - \angle CBK + \frac{\overset{\frown}{AK}}{2} = 180^\circ - \angle DBA + \angle ADK =$$
$$= \angle DBV + \angle BDV = 180^\circ - \angle AVD .$$

Consequaently $\angle CXA + \angle AVD = 180^\circ$. This equality means that the point X belongs to the circumcircle of ΔACV. For this reason $\angle XAB = \angle XCD$. The second equality could be obtained in the following way: $\angle XBC = \frac{\overset{\frown}{XC}}{2} = \angle XKC = 180^\circ - \angle AKX = \angle XDA$.

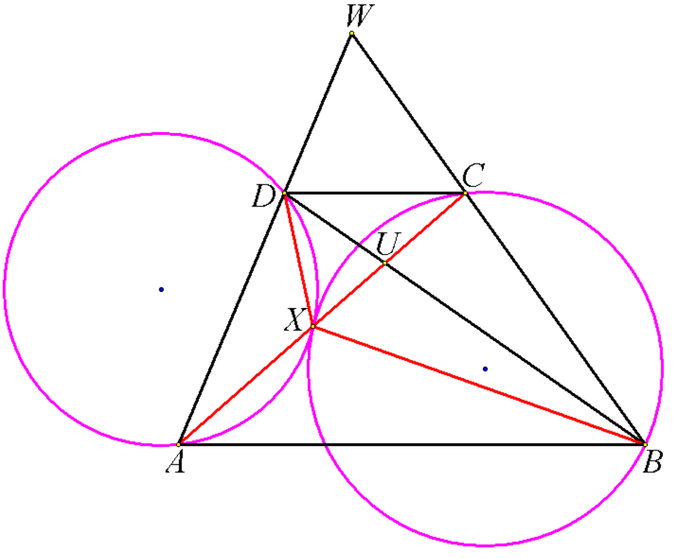
The assertion for the other couple of circles under the condition $AD \cap BC = W$ could be proven in the same way.





If $AB \parallel CD$ and the second common point of $k(BCK)$ and $k(DAK)$ is X, as in the case just considered, we obtain $\angle CXA = 180° - \angle DBA + \angle BDC = 180°$. This means that the points C, X and A are collinear, i.e. the point X is on the diagonal AC. Consequently, the common points X and K of the circles $k(BCK)$ and $k(DAK)$ are on the diagonal AC. But the circle $k(BCK)$ intersects the line AC in the point C. Since a circle and a line could have no more than two common points, then the point X coincides with K. Therefore, in this case the circles $k(BCK)$ and $k(DAK)$ are tangent at the point X.

Analogously, if $AD \parallel BC$, the circles $k(BCK')$ and $k(DAK')$ are tangent at the point $X \equiv K'$.



Solution of problem 6: If $ABCD$ is a parallelogram, then the equality $AB.CD = BC.DA$ is satisfied only in the case when $ABCD$ is a rhombus. Since the point X is the intersection point of its diagonals, then $\angle BXA + \angle DXC = 90° + 90° = 180°$. Let now $ABCD$ is such, that two of its sides at least are not parallel to each other. For definiteness let $AB \cap CD = V$. Since (according to lemma 5) the point X belongs to the circles $k(BCK)$ and $k(DAK)$, then $\angle DXA + \angle BXC = \angle AKD + \angle BKC$. From the conclusion we obtain, that $\angle DXA + \angle BXC = 180°$. But $(\angle BXA + \angle DXC) + (\angle DXA + \angle BXC) = 360°$. Therefore $\angle BXA + \angle DXC = 180°$. If we consider the point X as a common point of the circles $k(BCK')$ and $k(DAK')$, we deduce the desired equality as a consequence of lemma 5 and the conclusion in the following way:

$$\angle BXA + \angle DXC = \angle BK'A + \angle CK'D = 180°.$$

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ЗАДАЧА 6. ОТ IMO'2018

Резюме. Международната олимпиада по математика е едно от уважаваните събития и едно от най-старите международни образователни и научни състезания. Тя е най-голямата, най-старата и най-престижната научна олимпиада за гимназиални ученици. 59-ото издание на събитието се проведе в Клуж-Напока, Румъния, в периода 3 – 14 юли 2018. Настоящата статия е посветена на шестата задача от темата. Предложен е подробен анализ на задачата с методологически характер, който ще бъде полезен за ученици и учители по време на подготовката за бъдещи участия в математически състезания.

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Конкурсни задачи
Contest Problems
Рубриката се води от доц. д-р Веселин Ненков

РЕШЕНИЯ НА ЗАДАЧИТЕ ОТ БРОЙ 2, 2018

Задача 1. Да се намери най-малкото естествено число n, при което n куба с целочислени дължини на ръбовете в сантиметри имат сума на обемите, равна на $2018\,cm^3$.

Христо Лесов, Казанлък

Решение: тъй като $12^3 = 1728 < 2018 < 2197 = 13^3$, то 2018 не е куб на естествено число и затова $n > 1$. Разглеждаме последователно случаите за $n \geq 2$.

1) При $n = 2$ разглеждаме естествени числа x и y, за които са изпълнени релациите $x \geq y \geq 1$ и $x^3 + y^3 = 2018$. Тогава $2x^3 \geq x^3 + y^3 = 2018$, откъдето $x^3 \geq 1009 > 1000 = 10^3$, т.е. $x \geq 11$. Освен това $x^3 + 1 \leq x^3 + y^3 = 2018$, откъдето $x^3 \leq 2017 < 2197 = 13^3$, т.е. $x \leq 12$.Така получихме, че $11 \leq x \leq 12$. Лесно се проверява, че при $x = 11$ и $x = 12$ няма естествени стойности на y, за които е изпълнено равенството $x^3 + y^3 = 2018$.

2) При $n = 3$ разглеждаме естествени числа x, y и z, за които са изпълнени релациите $x \geq y \geq z \geq 1$ и $x^3 + y^3 + z^3 = 2018$. Тогава $3x^3 \geq x^3 + y^3 + z^3 = 2018$, откъдето $x^3 \geq 672 > 512 = 8^3$, т.е. $x \geq 9$. Освен това $x^3 + 1^3 + 1^3 \leq x^3 + y^3 + z^3 = 2018$, откъдето $x^3 \leq 2016 < 2197 = 13^3$, т.е. $x \leq 12$. Така получихме, че $9 \leq x \leq 12$. Ако $x = 9$ имаме $y^3 + z^3 = 1289$. Оттук $2y^3 \geq y^3 + z^3 = 1289$, $y^3 > 644 > 512 = 8^3$, т.е. $y \geq 9$. Освен това $y^3 + 1^3 \leq y^3 + z^3 = 1289$, $y^3 \leq 1288 < 1331 = 11^3$, т.е. $y \leq 10$. Така получихме, че $9 \leq y \leq 10$. Лесно се проверява, че при $x = 9$ и $y = 9,10$ няма естествени стойности на z, за които е изпълнено равенството $x^3 + y^3 + z^3 = 2018$. Ако $x = 10$, имаме $y^3 + z^3 = 1018$. Оттук $2y^3 \geq y^3 + z^3 = 1018$, $y^3 > 509 > 343 = 7^3$, т.е. $y \geq 8$. Освен това $y^3 + 1^3 \leq y^3 + z^3 = 1018$, $y^3 \leq 1017 < 1331 = 11^3$, т.е. $y \leq 10$. Така получихме, че $8 \leq y \leq 10$. Лесно се проверява, че при $x = 10$ и $y = 8,9,10$ няма естествени стойности на z, за които е изпълнено равенството $x^3 + y^3 + z^3 = 2018$. Ако $x = 10$, имаме $y^3 + z^3 = 1018$. Оттук $2y^3 \geq y^3 + z^3 = 1018$, $y^3 > 509 > 343 = 7^3$, т.е. $y \geq 8$. Освен това $y^3 + 1^3 \leq y^3 + z^3 = 1018$, $y^3 \leq 1017 < 1331 = 11^3$, т.е. $y \leq 10$. Така получихме, че $8 \leq y \leq 10$. Лесно се проверява, че при $x = 10$ и $y = 8,9,10$