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PARAMETRIC INTERACTION OF OPTICAL PULSES IN NONLINEAR ISOTROPIC MEDIUM

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Abstract. In present work the parametric interaction of two optical waves presented by their slowly varying amplitude functions $A_1(z)$ and $A_2(z)$ in isotropic nonlinear medium is investigated. The corresponding system of nonlinear amplitude differential equations describes the processes of third-harmonic generation and THz generation in femtosecond region. It is found analytical solution of this system, characterizing the periodic exchange of energy between the two waves. The obtained result can be used for investigating different parametric processes in nonlinear dielectric medium, depending on the value of the wavenumber mismatch *Dk*. These effects are directly related to modern communication systems and optical sensors.

Keywords: parametric interaction; narrow-band optical pulses; third-harmonic generation; four-wave mixing

Introduction

High harmonic generation (HHG) is a nonlinear effect that is able to produces light in visible and ultraviolet regions of the spectrum. In standard silica single mode fibers, the efficiency of Third-Harmonic Generation (THG) is typically low, because of the difficulties in the realization of phase-matching conditions. Special kinds of fibers are needed, e.g. SiO_2 waveguides doped with germanium, erbium and nitrogen. In this case the wave number mismatch Δk is low and THG is easily observed. There are different techniques for reducing wave number mismatch Δk presented by Agrawal (2007), Boyd (2007) and Dmitriev & Tarasov (1982). This phenomenon has been observed through the years in many experiments (Gabriagues, 1983); Österberg, 1990; Nicácio et al., 1993; Thøgersen & Mark,1994); Hickmann et al.,1995; De Araujo et al.,1996; Bufetov et al.,1997). The third-order susceptibility $\chi^{(3)}$ of the medium arises not only parametric THG effect but also leads to others nonlinear parametric effect such as Four-Wave Mixing (FWM). IF in addition in interaction between the waves we get in mind the carrier to envelope frequency, the wave number mismatch Δk becomes minimal and new spectral components with THz spectral shift (22 – 26 nm) to the short wavelengths, instead THG, are generated. Thus, by similar process the four-photon parametric mixing process between the waves generates coherent THz emission in glasses. In quasi-cw regime, when time derivatives can be set to zero, the corresponding system of evolutionary equations for THz generation is the same as that for THG one. We solve this system analytically. The obtained solutions, in the form of Jacobi elliptic functions, describe exchange of energy between the pulses.

Basic equation

The parametric evolution of two waves on different frequencies, for which the THG condition

$$\omega_2 = 3\omega_1, \tag{1}$$

or the THz generation conditions

$$\omega_2 = 3k_1(v_{ph} - v_{gr}), \tag{2}$$

where k_1 is the wave number, v_{ph} and v_{gr} are the phase and group velocity of the basic (A_1) wave, are satisfied can be govern by the following system of differential equations for their amplitude functions $A_1(z)$ and $A_2(z)$:

$$i\frac{\partial A_1}{\partial z} = \gamma_1 A_1^{*2} A_2 e^{-i\Delta kz};\ i\frac{\partial A_2}{\partial z} = \gamma_2 A_1^3 e^{i\Delta kz} \tag{3}$$

In the equations above g_1, g_2 and Dk are respectively nonlinearity of the medium for the two pulses and the wave number mismatch. In order to find a solution of the system of equation (3) we use a mathematical method similar to that described by Dmitriev & Tarasov (1982). We make the substitution below in order to find an analytical solution of system (3):

$$A_1(z) = a_1(z)e^{i\varphi_1(z)};\ A_2(z) = a_2(z)e^{i\varphi_2(z)} \tag{4}$$

where $a_1(z)=|A_1(z)|$, $a_2(z)=|A_2(z)|$, $j_1(z)$ and $j_2(z)$ are real functions, describing the amplitude functions and phases of the two pulses. By replacing Eq. (4) in Eq. (3), the system of differential equations can be presented in the form of:

$$i\frac{\partial a_1}{\partial z} - a_1\frac{\partial \varphi_1}{\partial z} = \gamma_1 a_1^2 a_2 e^{-i\Psi(z)};\ i\frac{\partial a_2}{\partial z} - a_2\frac{\partial \varphi_2}{\partial z} = \gamma_2 a_1^3 e^{i\Psi(z)}, \tag{5}$$

where the generalized phase is of the kind:

$$\Psi(z) = 3\varphi_1 - \varphi_2 + \Delta kz \tag{6}$$

As a next step, we equalize the real and imaginary parts on both sides of the equalities. The following systems of ordinary differential equations are obtained:

$$Re:\quad \frac{\partial \varphi_1}{\partial z} = -\gamma_1 a_1 a_2 cos\Psi \tag{7a}$$

$$\frac{\partial \varphi_2}{\partial z} = -\gamma_2 \frac{a_1^3}{a_2} cos\Psi \tag{7b}$$

$$Im:\quad \frac{\partial a_1}{\partial z} = -\gamma_1 a_1^2 a_2 sin\Psi \tag{8a}$$

$$\frac{\partial a_2}{\partial z} = \gamma_2 a_1^3 sin\Psi \tag{8b}$$

Let us consider first the system of Eqs. (8a) and (8b), obtained by equalizing the imaginary parts of Eq. (5). It can be presented in the form of one equation as follow: we multiply the first Eq. (8a) by $\gamma_2 a_1$ and the second Eq. (8b) by $\gamma_1 a_2$. Moreover, we collect the obtained expressions. The result is presented as follow:

$$\frac{\gamma_2}{2}\frac{\partial(a_1^2)}{\partial z} + \frac{\gamma_1}{2}\frac{\partial(a_2^2)}{\partial z} = 0 \tag{9}$$

where a_1^2 and a_2^2 are the intensities of the two pulses. Eq. (9) can be written as well in the form:

$$\frac{1}{2}\frac{\partial}{\partial z}[\gamma_2 a_1^2 + \gamma_1 a_2^2] = 0 \tag{10}$$

As it can be clearly seen:

$$\gamma_2 a_1^2 + \gamma_1 a_2^2 = C = const. \tag{11}$$

The Eq. (11) presents the conservation law for the sum of the intensities of two pulses. As a next step, we will consider the system of Eqs. (7a) and (7b), obtained by equalizing the real parts of Eq. (5).

We multiply the first Eq.(7a) by $\gamma_2 a_1^2$ and the second Eq. (7b) by $\gamma_1 a_2^2$. Thus, we deduct the two expressions. The result is presented as follow.

$$\gamma_2 a_1^2 \frac{\partial \varphi_1}{\partial z} - \gamma_1 a_2^2 \frac{\partial \varphi_2}{\partial z} \tag{12}$$

Having in mind the conservation law for the sum of the intensities of two pulses, Eq. (12) can be presented in the form:

$$\frac{\partial \varphi_1}{\partial z} = \frac{\gamma_1 a_2^2}{C - \gamma_1 a_2^2}\frac{\partial \varphi_2}{\partial z} \tag{13}$$

The expression above gives the connection between the derivatives of the pulses' phases. Let us now go back to the expression for the generalized Eq. (6). By using it, we can find a link between functions, characterizing the generalized phrase Y and the intensity of one of the pulses a_2. We differentiate Eq. (6) by the independent variable z. As a result, the next equation is obtained:

$$\frac{\mathrm{d}\Psi(z)}{dz} = 3\frac{d\varphi_1}{dz} - \frac{d\varphi_2}{dz} + \Delta k \tag{14}$$

In the equation above if we substitute the derivatives $\frac{d\varphi_1}{dz}$ and $\frac{d\varphi_2}{dz}$ respectively with expressions (7a) and (7b), Eq.(14) can be presented in the form of:

$$\frac{\mathrm{d}\Psi(z)}{dz} = \frac{a_1}{a_2}[\gamma_2 a_1^2 - 3\gamma_1 a_2^2]cos\Psi + \Delta k \tag{15}$$

The unknown functions in this equation are Y, a_1 and a_2. However, there is a connection between the functions a_1 and a_2, given by Eq. (11). Thus, it is found that:

$$a_1 = \frac{1}{\sqrt{\gamma_2}}\sqrt{C - \gamma_1 a_2^2} \tag{16}$$

By substituting Eq. (16) in Eq. (15), it is obtained an equation, in which are included the unknown functions Y and a_2:

$$\frac{\mathrm{d}\Psi(z)}{dz} = \frac{1}{a_2\sqrt{\gamma_2}}\sqrt{C - \gamma_1 a_2^2}[C - 4\gamma_1 a_2^2]cos\Psi + \Delta k \tag{17}$$

For this reason, by using Eqs. (17), (16) and (8b), it is obtained the following system:

$$\frac{\mathrm{d}\Psi(z)}{dz} = \frac{1}{a_2\sqrt{\gamma_2}}\sqrt{C - \gamma_1 a_2^2}[C - 4\gamma_1 a_2^2]cos\Psi + \Delta k \tag{18a}$$

$$\frac{\partial a_1}{\partial z} = \frac{[C - \gamma_1 a_2^2]^{3/2}}{\sqrt{\gamma_2}}sin\Psi \tag{18b}$$

This system of equations can be reduced to one by multiplying Eq. (18a) with *sin*Y and divide Eq. (18a) to Eq. (18b)

$$\frac{d\Psi(z)}{da_2} = -\frac{[C-4\gamma_1 a_2^2]}{a_2[C-\gamma_1 a_2^2]}cos\Psi - \frac{\Delta k\sqrt{\gamma_2}}{[C-4\gamma_1 a_2^2]^{3/2}} \tag{19}$$

The equation above is a linear ordinary differential equation of first order, in which the unknown function is *cos*Y and the unknown variable is a_2. It can be solved by using well-known methods. Its solution is of the kind:

$$cos\Psi = \frac{2B - \Delta k a_2^2\sqrt{\gamma_2}}{2a_2[C-\gamma_1 a_2^2]^{3/2}} \tag{20}$$

where *B* is an integration constant. The Eq. (20) can be written in the form:

$$2a_2[C-\gamma_1 a_2^2]^{3/2}cos\Psi + \Delta k a_2^2\sqrt{\gamma_2} = 2B = const \tag{21}$$

The expression (21) gives the connection between intensity a_2 and the generalized phase Y for arbitrary value of the variable *z*. It can be considered as conservation law for these quantity. Thus, we have found a connection between a_1 and a_2 (Eq. (11)), the derivatives $\frac{d\varphi_1}{dz}$ and $\frac{d\varphi_2}{dz}$ (Eq. (13)) and intensity a_2 and the generalized phrase Y (Eq. (21)). By using these results, our next step will be defining the form of the functions: $a_2^2 = a_2^2(z), a_1^2 = a_1^2(z), \varphi_1 = \varphi_1(z)$ and $\varphi_2 = \varphi_2(z)$, characterizing respectively the intensities and phases of the pulses.

Let us first consider Eq. (8b), describing the behavior of the amplitude function $a_2^2(z)$ of the second pulse. In the expression it is included *sin*Y. Having in mind Eqs. (16) and (20) and that $sin\Psi = \sqrt{1-cos^2\Psi}$, equation (8b) can be presented as follow:

$$\frac{\partial a_2}{\partial z} = \frac{[C-\gamma_1 a_2^2]^{3/2}}{\sqrt{\gamma_2}}\sqrt{1 - \frac{(2B-\Delta k a_2^2\sqrt{\gamma_2})^2}{4a_2^2[C-\gamma_1 a_2^2]^3}} \tag{22}$$

It is useful in the equation above to make the following substitution:

$$a_2^2(z) = p(z) \tag{23}$$

After a couple of transformations, it can be presented in the form:

$$\frac{dp}{dz} = 2\gamma_1 \sqrt{\frac{\gamma_1}{\gamma_2}} \sqrt{p(\alpha - p)^3 - \eta(\beta - p)^2} \quad (24)$$

where for $\Delta k \neq 0$:

$$\alpha = \frac{C}{\gamma_1} = const > 0, \beta = \frac{2B}{\Delta k \sqrt{\gamma_2}} = const > 0, \eta = \frac{\gamma_2 \Delta k^2}{4\gamma_1^3} \quad (25)$$

As a next step, in order to remove the coefficient in front of the square root on the right side, we change the variable z with ξ by the substitution:

$$\xi = 2\gamma_1 \sqrt{\frac{\gamma_1}{\gamma_2}} z \quad (26)$$

The function on the right side of Eq. (24) is complicated - fourth degree polynomial of p, in which are included all degrees:

$$F(p) = p(\alpha - p)^3 - \eta(\beta - p)^2 = -p^4 + 3\alpha p^2 - (3\alpha^2 + \eta)p^2 - \eta\beta^2 \quad (27)$$

Using the substitution:

$$p = y + \frac{3}{4}\alpha, \quad (28)$$

where y is new unknown function; the function under square root can be presented as follow:

$$F(y) = \bar{A}y^4 + \bar{B}y^2 + \bar{C}y + \bar{D} \quad (29)$$

where

$$\bar{A} = -1\bar{A} = -1, \quad \bar{B} = \frac{3}{2}\left(\frac{\alpha}{2}\right)^2 - \eta = const,$$
$$\bar{C} = 2\beta\eta - 3\eta\left(\frac{\alpha}{2}\right) - \left(\frac{\alpha}{2}\right)^3 = const, \quad (30)$$
$$\bar{D} = \frac{3}{16}\left(\frac{\alpha}{2}\right)^4 - \frac{9}{4}\left(\frac{\alpha}{2}\right)^2 + 3\beta\eta\left(\frac{\alpha}{2}\right) - \eta\beta^2 = const$$

We replace Eqs. (28), (29) and (30) in Eq. (24). As result it is written in the form:

$$\frac{dy}{d\xi} = \sqrt{\bar{A}y^4 + \bar{B}y^2 + \bar{C}y + \bar{D}} \tag{31}$$

We raise to power 2 the both sides of the equation above. As a next step we differentiate it by the unknown variable $\xi. \xi.$ After a couple of transformations it can be presented as follow:

$$\frac{d^2y}{d\xi^2} - 2\bar{A}y^3 - \bar{B}y - \frac{\bar{C}}{2} = 0 \tag{32}$$

Eq. (32) is of the same kind as the following equation:

$$\frac{d^2y}{d\xi^2} - (2 - \bar{\kappa}^2)y + 2y^3 = 0, \quad 0 < \bar{\kappa} < 1 \tag{33}$$

Under the initial condition *y(0)=1*, the solution of Eq. (33) is the elliptic delta function. Having in mind the Eq. (30) for the constants and making the substitution:

$$\bar{C} = 0, \tag{34}$$

Eq. (32) can be presented in the form:

$$\frac{d^2y}{d\xi^2} - \left(\frac{3}{2}\left(\frac{\alpha}{2}\right)^2 - \eta\right)y + 2y^3 = 0 \tag{35}$$

When

$$\bar{\kappa}^2 = 2 + \eta - \frac{3}{2}\left(\frac{\alpha}{2}\right)^2 \tag{36}$$

Eq. (35) takes exactly the form of Eq. (33).

From the assumptions made by now, it is clearly seen that under condition $(\bar{C} = 0)$

$$2\beta\eta - 3\eta\left(\frac{\alpha}{2}\right) - \left(\frac{\alpha}{2}\right)^3 = 0;\ \bar{\kappa} = \sqrt{2 + \eta - \frac{3}{2}\left(\frac{\alpha}{2}\right)^2} < 1 \qquad (37)$$

By using Eqs. (23), (28) and (11) we find that:

$$a_2^2 = \frac{3}{4}\frac{C}{\gamma_1} + dn(\xi, \bar{\kappa}) \qquad (38)$$

$$a_1^2 = \frac{1}{4}\frac{C}{\gamma_2} - \frac{\gamma_1}{\gamma_2} dn(\xi, \bar{\kappa}) \qquad (39)$$

The expressions above characterize the intensities of two pulses during their evolution in nonlinear isotropic media.

Conclusions

In the paper, it is presented a mathematical model describing the process of nonlinear interaction of two waves satisfied the THG (Eq, (1)) and THz generation (Eq. (2)) phase mismatch conditions. When the second order of dispersion is neglected the system of shortcut Eq. (3) can be analytically solved. The obtained solutions, in the form of Jacobi elliptic functions, describe the periodic exchange of energy between the pulses.

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