ONE TYPE OF PROBLEMS WITH INFINITE NUMBER OF CUBE ROOTS

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Abstract. In the present research paper a particular type of problems is examined, with an infinite number of cube roots involved. Different problems and questions are proposed and in terms of solving some examples two theorems are defined and proved.

The study could be a good base for generating various problems from this type, used in mathematical competitions, Olympiads and for preparing such events.

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In math competitions and Olympiads, it is a common situation one type of interesting problems to be present. These problems include square roots and they are usually solved by number sequences in recurrence relation. (Nikolaev, Milkova & Petkov, 2017a, b). We propose a similar example but the square roots are substituted by cube roots:

Example 1. Determine the value of the expression:

$$I = \sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\dots}}}\ .$$

Solution. Let us consider the following sequence in recurrence relation $\{a_n\}_{n=1}^{\infty}$. We have

$$a_1 = \sqrt[3]{6},\ a_n = \sqrt[3]{6+a_{n-1}}\ \ for\ n = 2,3,\dots.$$

Hence $I = \lim\limits_{n\to\infty} a_n$, if this limit does exist.

Firstly, we will prove that 2 is an upper bound for the sequence $\{a_n\}_{n=1}^{\infty}$.

We will apply mathematical induction:

1. For $n = 1$, it is obvious that $a_1 = \sqrt[3]{6} < 2$.
2. We assume that the statement holds for $n = k$, i.e. $a_k < 2$.
3. We will prove that it holds for $n = k+1$:

$a_{k+1} = \sqrt[3]{6+a_k} < \sqrt[3]{6+2} = 2$, because $a_k < 2$ from the previous point.

So, this is the proof that the sequence $\{a_n\}_{n=1}^{\infty}$ has 2 as an upper bound.

Secondly, we will prove that the sequence $\{a_n\}_{n=1}^{\infty}$ is strictly monotonic increasing.

Let us assume the opposite: $a_{n+1} = \sqrt[3]{6+a_n} \le a_n$.

$$a_{n+1} = \sqrt[3]{6+a_n} \le a_n \Big|^3 \Leftrightarrow a_n^3 - a_n - 6 \ge 0 \Leftrightarrow a_n^3 - 8 - (a_n - 2) \ge 0 \Leftrightarrow$$
$$\Leftrightarrow (a_n - 2)(a_n^2 + 2a_n + 4) - (a_n - 2) \ge 0 \Leftrightarrow (a_n - 2)(a_n^2 + 2a_n + 3) \ge 0.$$

But $a_n^2 + 2a_n + 3 = (a_n + 1)^2 + 2 > 0$ and $a_n - 2 < 0$ $(a_n < 2)$, so the assumption is impossible and $a_{n+1} > a_n$.

Hence $\{a_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence with an upper bound 2, which leads us to the conclusion that this is a convergent sequence. If so, we can assume that its limit is l. Consequently:

$$a_{n+1} = \sqrt[3]{6+a_n}, \quad \lim_{n\to\infty} a_{n+1} = \sqrt[3]{6+\lim_{n\to\infty} a_n}, \quad l^3 = 6 + l, \quad l^3 - l - 6 = 0 \Rightarrow$$

$l^3 - 2^3 - l + 2 = 0 \Rightarrow (l-2).(l^2 + 2l + 3) = 0 \Rightarrow$ the equation has only one real root $l = 2 \Rightarrow I = 2$.

Let us formalize the following:

Theorem 1. $\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}} = d$ for $d^2 \in [k; 4k)$ and $d > 0$.

Proof. If $d^2 = k \Rightarrow \sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}} = \sqrt[3]{d^3} = d$.

For $d^2 \in (k; 4k)$ let us examine the sequence $\{a_n\}_{n=1}^{\infty}$, such that:

$a_1 = \sqrt[3]{kd}$, $a_n = \sqrt[3]{kd + (d^2 - k)a_{n-1}}$, $n = 2, 3, \ldots$.

Firstly, we will prove that d is an upper bound for the sequence $\{a_n\}_{n=1}^{\infty}$.

We will apply mathematical induction:

1. $n = 1$. Let us check if $a_1 = \sqrt[3]{kd} < d$.

$a_1 = \sqrt[3]{kd} < d \Big|^3 \Leftrightarrow kd < d^3 \Big| : d > 0 \Leftrightarrow k < d^2$ – correct according to $d^2 \in (k; 4k)$.

2. We assume that the statement holds for $n = k$, i.e. $a_k < d$.

3. We will prove that it holds for $n = k+1$:

$a_{k+1} = \sqrt[3]{dk + (d^2 - k)a_k} < \sqrt[3]{dk + (d^2 - k)d} = \sqrt[3]{d^3} = d$, because $a_k < d$ from the previous point.

So, this is the proof that the sequence $\{a_n\}_{n=1}^{\infty}$ has d as an upper bound.

Secondly, we will prove that the sequence $\{a_n\}_{n=1}^{\infty}$ is strictly monotonic increasing.

Let us assume the opposite: $a_{n+1} = \sqrt[3]{dk + (d^2 - k)a_n} \le a_n$.

$$\sqrt[3]{dk + (d^2 - k)a_n} \le a_n \,\Big|^3 \Leftrightarrow a_n^3 - (d^2 - k)a_n - dk \ge 0 \Leftrightarrow a_n^3 - d^3 - (d^2 - k)a_n - dk + d^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (a_n - d)(a_n^2 + da_n + d^2) - (d^2 - k)(a_n - d) \ge 0 \Leftrightarrow (a_n - d)(a_n^2 + da_n + k) \ge 0.$$

For the expression $a_n^2 + da_n + k$ we have $D = d^2 - 4k < 0 \left(\text{from } d^2 \in (k;4k)\right)$. Consequently, the expression $a_n^2 + da_n + k$ will be always positive.

Taking into consideration that $a_n - d < 0 \left(\text{from } a_n < d\right)$, the assumption is impossible and thus we proved that $a_{n+1} > a_n$.

Hence $\{a_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence with an upper bound d, which leads us to the conclusion that this is a convergent sequence. If so, we can assume that its limit is l. Consequently:

$$a_{n+1} = \sqrt[3]{dk + (d^2 - k)a_n} \ , \ \lim_{n\to\infty} a_{n+1} = \sqrt[3]{dk + (d^2 - k)\lim_{n\to\infty} a_n} \ ,$$

$l^3 = dk + (d^2 - k)l$. The last equation has only one real root ($l = d$), because it is equivalent to $(l - d).(l^2 + dl + k) = 0$, and the discriminant of the second multiplier is negative $(d^2 - 4k < 0)$.

Thus, we proved that $\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}} = d$ for $d^2 \in [k;4k)$ and $d > 0$.

In this statement d and k are positive numbers. Applying the theorem, we can solve the following:

Example 2. Determine the value of the expression:

$$\sqrt[3]{55 + 14\sqrt[3]{55 + 14\sqrt[3]{55 + \ldots}}} \ .$$

Solution.

$$\begin{cases} 55 = kd \\ 14 = d^2 - k \end{cases} \Leftrightarrow \begin{cases} k = \dfrac{55}{d} \\ 14 = d^2 - \dfrac{55}{d} \end{cases} \Leftrightarrow \begin{cases} k = \dfrac{55}{d} \\ d^3 - 14d - 55 = 0 \end{cases};$$

$d = 5,\ k = 11,\ d^2 = 25 \in (11;44)$.

Consequently, from the theorem we will have $\Rightarrow \sqrt[3]{55 + 14\sqrt[3]{55 + 14\sqrt[3]{55 + \ldots}}} = 5$.

Based on Theorem 1 a number of competition problems can be composed.

If $d = 4,\ d^2 = 16$, then k can be each number from the interval $(5;15)$.

The values $k = 7$ and $d = 4$ lead to the following:

Example 3. Prove that $\sqrt[3]{28 + 9\sqrt[3]{28 + 9\sqrt[3]{28 + \ldots}}} = 4$.

The next examples are solved by applying the theorem immediately, following its constraints with various real values for k and d.

Example 4. Determine the value of the expression $\sqrt[3]{15 + 4\sqrt[3]{15 + 4\sqrt[3]{15 + \ldots}}}$.

Example 5. Determine the value of the expression $\sqrt[3]{9 + 6\sqrt[3]{9 + 6\sqrt[3]{9 + \ldots}}}$.

Example 6. Determine the value of the expression $\sqrt[3]{12 + 5\sqrt[3]{12 + 5\sqrt[3]{12 + \ldots}}}$.

Example 7. Determine the value of the expression $\sqrt[3]{20 + 11\sqrt[3]{20 + 11\sqrt[3]{20 + \ldots}}}$.

Example 8. Determine the value of the expression $\sqrt[3]{14 + 8,25\sqrt[3]{14 + 8,25\sqrt[3]{14 + \ldots}}}$.

Example 9. Determine the value of the expression $\sqrt[3]{2\sqrt{5} + 3\sqrt[3]{2\sqrt{5} + 3\sqrt[3]{2\sqrt{5} + \ldots}}}$.

Example 10. Determine the value of the expression $\sqrt[3]{6 + (6 - \sqrt{6})\sqrt[3]{6 + (6 - \sqrt{6})\sqrt[3]{6 + \ldots}}}$.

Since the sign does not matter in the odd rootfunctions, we will prove the following:

Theorem 2. $\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}} = d$ for $d^2 \in [k;4k)$ and $d < 0$.

Proof. For $d^2 = k \Rightarrow \sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}} = \sqrt[3]{d^3} = d$.

If $d^2 \in (k;4k)$ we can examine the sequence $\{a_n\}_{n=1}^{\infty}$, where

$a_1 = \sqrt[3]{kd},\ a_n = \sqrt[3]{kd + (d^2 - k)a_{n-1}}$ for $n = 2,3,\ldots$.

Firstly, we will prove that d is a lower bound for the sequence $\{a_n\}_{n=1}^{\infty}$.

We will apply mathematical induction:

1. $n = 1$. Let us check if $a_1 = \sqrt[3]{kd} > d$.

$a_1 = \sqrt[3]{kd} > d \Big|^3 \Leftrightarrow kd > d^3 \Big| : d < 0 \Leftrightarrow k < d^2$ – correct according to $d^2 \in (k; 4k)$.

2. We assume that the statement holds for $n = k$, i.e. $a_k > d$.

3. We will prove that it holds for $n = k + 1$:

$a_{k+1} = \sqrt[3]{dk + \left(d^2 - k\right)a_k} > \sqrt[3]{dk + \left(d^2 - k\right)d} = \sqrt[3]{d^3} = d$, because $a_k > d$ from the previous point.

So, this is the proof that the sequence $\left\{a_n\right\}_{n=1}^{\infty}$ has d as a lower bound.

Secondly, we will prove that the sequence $\left\{a_n\right\}_{n=1}^{\infty}$ is strictly monotonic decreasing.

Let us assume the opposite: $a_{n+1} = \sqrt[3]{dk + \left(d^2 - k\right)a_n} \geq a_n$.

$\sqrt[3]{dk + \left(d^2 - k\right)a_n} \geq a_n \Big|^3 \Leftrightarrow a_n^3 - \left(d^2 - k\right)a_n - dk \leq 0 \Leftrightarrow a_n^3 - d^3 - \left(d^2 - k\right)a_n - dk + d^3 \leq 0 \Leftrightarrow$

$\Leftrightarrow \left(a_n - d\right)\left(a_n^2 + da_n + d^2\right) - \left(d^2 - k\right)\left(a_n - d\right) \leq 0 \Leftrightarrow \left(a_n - d\right)\left(a_n^2 + da_n + k\right) \leq 0.$

The discriminant of the expression $a_n^2 + da_n + k$ is $D = d^2 - 4k < 0$ $\left(\text{from } d^2 \in \left(k; 4k\right)\right)$. Consequently, the expression $a_n^2 + da_n + k$ is always positive.

Since $a_n > d$, it is obvious that the assumption is impossible and $a_{n+1} < a_n$.

Hence $\left\{a_n\right\}_{n=1}^{\infty}$ is a monotonic decreasing sequence with a lower bound d, which leads us to the conclusion that this is a convergent sequence. If so, we can assume that its limit is l. Consequently:

$$a_{n+1} = \sqrt[3]{dk + \left(d^2 - k\right)a_n}, \quad \lim_{n\to\infty} a_{n+1} = \sqrt[3]{dk + \left(d^2 - k\right)\lim_{n\to\infty} a_n},$$

$l^3 = dk + \left(d^2 - k\right)l$. The last equation has only one real root $\left(l = d\right)$, because it is equivalent to $\left(l - d\right).\left(l^2 + dl + k\right) = 0$, and the discriminant of the second multiplier is negative $\left(d^2 - 4k < 0\right)$.

Thus, we proved that $\sqrt[3]{kd + \left(d^2 - k\right)\sqrt[3]{kd + \left(d^2 - k\right)\sqrt[3]{kd + \ldots}}} = d$

for $d^2 \in \left[k; 4k\right)$ and $d < 0$.

From Theorem 1 and Theorem 2 we can conclude that for all real numbers $d \neq 0$ and k, such that $d^2 \in [k;4k)$, the expression $\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + (d^2 - k)\sqrt[3]{kd + \ldots}}}$ is equal to d.

These theorems enable us to enlarge the range of competition problems with infinite number of cube roots.

Example 11. Determine the value of the expression $\sqrt[3]{-4 + 2\sqrt[3]{-4 + 2\sqrt[3]{-4 + \ldots}}}$.

Example 12. Determine the value of the expression $\sqrt[3]{-12 + 5\sqrt[3]{-12 + 5\sqrt[3]{-12 + \ldots}}}$.

Example 13. Determine the value of the expression $\sqrt[3]{-9 + 6\sqrt[3]{-9 + 6\sqrt[3]{-9 + \ldots}}}$.

Both theorems and the examples that were given offer some different options to compose and solve various math problems for competitions and Olympiads. The authors hope that this material may improve the preparation of the participants and may give some additional tools that they can use.

REFERENCES

Nikolaev, R., Milkova, T. & Petkov, Y. (2017 a). Some Numerical Sequences Concerning Square Roots (Part One), *Mathematics and Informatics*, 5 (60), p.474 – 480.

Nikolaev, R., Milkova, T. & Petkov, Y. (2017 b). Some Numerical Sequences Concerning Square Roots (Part Two), *Mathematics and Informatics*, 6 (60), p.616 – 625.

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