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# ONE MORE PROOF FOR THE DISTANCE BETWEEN THE INCENTRE AND THE ORTHOCENTRE OF THE TRIANGLE

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**Abstract**. One more proof is proposed in the present paper concerning the the distance between the incentre ($I$) and the orthocentre ($H$) of the triangle.

*Keywords*: distance, incentre, orthocentre, sine law, cosine law, centroid of a triangle.

The following equality is considered in the paper:

$$|IH|^2 = 2r^2 - 4R^2 \cos\alpha \cos\beta \cos\gamma, \tag{1}$$

where $r$ and $R$ are respectively the radii of the incircle and the circumcircle of the triangle $\Delta ABC$ with angles $\alpha$, $\beta$ and $\gamma$. One proof of the equality, which is not simple at all, is propesed by the present author in his paper (Arslanagić, 2005, pp. 437-439). For a new proof the next two equalities will be used:

$$|IO|^2 = R^2 - 2Rr, \tag{2}$$

where $O$ is the circumcentre of the triangle and

$$|OH|^2 = 9r^2 - \left(a^2 + b^2 + c^2\right), \tag{3}$$

where $a$, $b$, $c$ are the lengths of its sides.

Two proofs of the equality (2) are proposed by the present author in (Arslanagić, 2005, pp. 432-434) and one my proof in (Arslanagić, 2009, pp. 79-80). Similarly, two proofs of the equality (3) are proposed in (Arslanagić, 2005, pp. 435-437).

One more equality will be used in the sequel:

$$|IT|^2 = \frac{2}{3}r^2 - \frac{4}{3}Rr + \frac{1}{18}\left(a^2 + b^2 + c^2\right), \tag{4}$$

where $T$ is the centroid of the triangle $\Delta ABC$. Here is a proof of (4):

*Proof*: The idea is to apply Leibniz theorem for the triangle $\Delta ABC$, see (Arslanagić, 2005, pp. 354-355) or (Grozdev, 2007), i.e.:

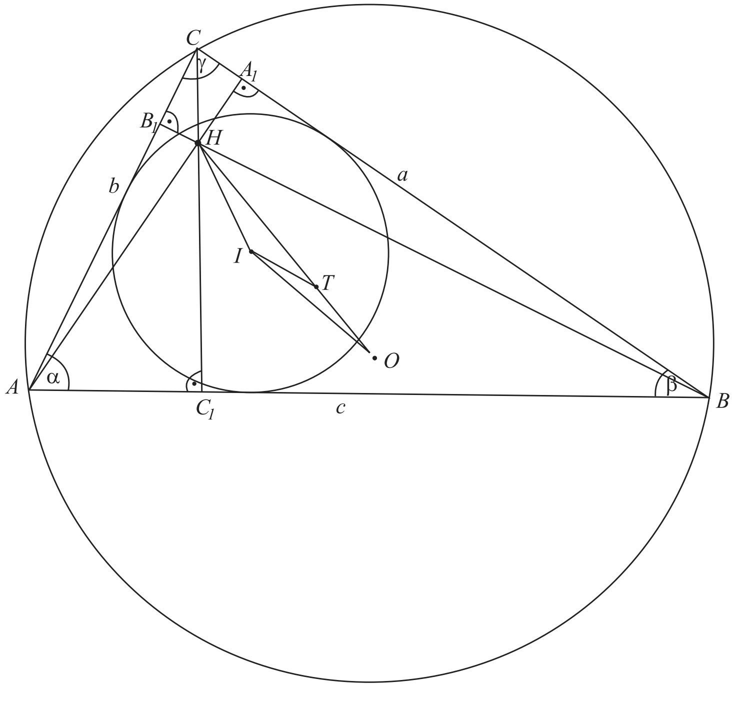
$$|MA|^2+|MB|^2+|MC|^2=3|MT|^2+|AT|^2+|BT|^2+|CT|^2, \tag{5}$$

where the point $M$ is not necessary in the plane of the triangle $\Delta ABC$. Since $|AT|=\frac{2}{3}m_a$, $|BT|=\frac{2}{3}m_b$, $|CT|=\frac{2}{3}m_c$, ( $m_a, m_b, m_c$ are the medians of the traingle $\Delta ABC$), we have:

$$|AT|^2+|BT|^2+|CT|^2=\frac{4}{9}\left(m_a^2+m_b^2+m_c^2\right)=\frac{4}{9}\cdot\frac{3}{4}\left(a^2+b^2+c^2\right)=\frac{1}{3}\left(a^2+b^2+c^2\right).$$

By the substitutions $|MA|=x, |MB|=y, |MC|=z$ we get:

$$9|MT|^2=3\left(x^2+y^2+z^2\right)-\left(a^2+b^2+c^2\right).$$




If $M$ is replaced by $I$, then:

$$x^2=(s-a)^2+r^2,\ y^2=(s-b)^2+r^2,\ z^2=(s-c)^2+r^2;\ \left(s=\frac{1}{2}(a+b+c)\right)$$

$$9|IT|^2=3\left[(s-a)^2+r^2+(s-b)^2+r^2+(s-c)^2+r^2\right]-\left(a^2+b^2+c^2\right)$$

$$\Rightarrow 9|IT|^2=3\left[\left(3s^2+2s(a+b+c)+a^2+b^2+c^2+3r^2\right)\right]-\left(a^2+b^2+c^2\right)$$

$$\Rightarrow 9|IT|^2=9r^2-3s^2+2\left(a^2+b^2+c^2\right). \tag{6}$$

Since $a^2+b^2+c^2=2s^2-2r^2-8Rr$, then.

$$2s^2=a^2+b^2+c^2+2r^2+8Rr$$

$$\Rightarrow s^2=\frac{1}{2}\left(a^2+b^2+c^2\right)+r^2+4Rr. \tag{7}$$

From (6) and (7) we get:

$$9|IT|^2=9r^2-\frac{3}{2}\left(a^2+b^2+c^2\right)-3r^2-12Rr+2\left(a^2+b^2+c^2\right)$$

$$\Rightarrow |IT|^2=\frac{2}{3}r^2-\frac{4}{3}Rr+\frac{1}{18}\left(a^2+b^2+c^2\right),$$

and this is the equality (4).

Further, we give the proof of the equality (1).

By the cosine law for the triangle $\Delta TIO$:

$$|IT|^2=|IO|^2+|OT|^2-2|IO|\cdot|OT|\cos\varphi, \tag{8}$$

where $\varphi=\measuredangle IOT$.

It is well known that the points $O, T$ and $H$ are collinear, lying on the **Euler's line**. We will use the corresponding **Euler's theorem**:

$$|OT|=\frac{1}{3}|OH|. \tag{9}$$

From (8) and (9) it follows, that

$$\cos\varphi=\frac{|IO|^2+|OT|^2-|IT|^2}{2|IO|\cdot|OT|}$$

$$\Rightarrow cos\,\varphi = \frac{|IO|^2 + \frac{1}{9}|OH|^2 - |IT|^2}{2|IO|\cdot\frac{1}{3}|OH|}$$

$$\Rightarrow cos\,\varphi = \frac{9|IO|^2 + |OH|^2 - 9|IT|^2}{6|IO|\cdot|OH|}\,. \tag{10}$$

Apply now the cosine law for the triangle $\Delta IHO$:

$$|IH|^2 = |IO|^2 + |OH|^2 - 2|IO|\cdot|OH|\,cos\,\varphi\,,$$

and from here, because of (10), we have:

$$|IH|^2 = |IO|^2 + |OH|^2 - 2|IO|\cdot|OH|\cdot\frac{9|IO|^2 + |OH|^2 - 9|IT|^2}{6|IO|\cdot|OH|}$$

$$\Rightarrow |IH|^2 = |IO|^2 + |OH|^2 - 3|IO|^2 - \frac{1}{3}|OH|^2 + 3|IT|^2$$

$$\Rightarrow |IH|^2 = -2|IO|^2 + \frac{2}{3}|OH|^2 + 3|IT|^2\,. \tag{11}$$

Finally, from (11) on the base of (2), (3) and (4), we obtain:

$$|IH|^2 = -2R^2 + 4Rr + 6R^2 - \frac{2}{3}\left(a^2+b^2+c^2\right) + 2r^2 - 4Rr + \frac{1}{6}\left(a^2+b^2+c^2\right)$$

$$\Rightarrow |IH|^2 = 4R^2 + 2r^2 - \frac{1}{2}\left(a^2+b^2+c^2\right),$$

next by the sine law:

$$\Rightarrow |IH|^2 = 4R^2 + 2r - \frac{1}{2}\cdot 4R^2\left(sin^2\alpha + sin^2\beta + sin^2\gamma\right),$$

and by the well known identity

$$sin^2\alpha + sin^2\beta + sin^2\gamma = 2 + 2\,cos\,\alpha\,cos\,\beta\,cos\,\gamma$$

we have:

$$|IH|^2 = 4R^2 + 2r^2 - 2R^2\left(2 + 2\,cos\,\alpha\,cos\,\beta\,cos\,\gamma\right), \text{ i.e.}$$

$$|IH|^2 = 2r^2 - 4R^2\,cos\,\alpha\,cos\,\beta\,cos\,\gamma\,.$$

**Corollary**. Prove the inequality 2.23 from (Bottema & al., 1969, p. 25), i.e.

$$\cos\alpha\cos\beta\cos\gamma \le \frac{1}{8}, \tag{12}$$

where the equality holds true if and only if $\alpha = \beta = \gamma$, i.e. if the triangle is equilateral.

*Proof*: Since $|IH|^2 \ge 0$, it follows from (1), that:

$$2r^2 - 4R^2 \cos\alpha\cos\beta\cos\gamma \ge 0$$

$$\Rightarrow \cos\alpha\cos\beta\cos\gamma \le \frac{r^2}{2R^2}. \tag{13}$$

It is well known the inequality $R \ge 2r$ (it follows from (2): $|IO|^2 \ge 0 \Rightarrow R^2 - 2Rr \ge 0 \Rightarrow R \ge 2r$). Thus, $\frac{r}{R} \le \frac{1}{2} \Rightarrow \frac{r^2}{R^2} \le \frac{1}{4} \Rightarrow \frac{r^2}{2R^2} \le \frac{1}{8}$ and we conclude that the inequality (13) is stronger than the inequality (12). The equality in (13) holds true for the equilateral triangle too.

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## ТРИ РЕШЕНИЯ НА ЕДНА ЗАДАЧА С ЧЕТИРИ ОКРЪЖНОСТИ

**Резюме.** Статията е посветена на три решения на една интересна задача за четири окръжности.

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