

MULTIPLE REPRESENTATIONS OF FUNCTIONS IN THE FRAME OF DISTANCE LEARNING

Radoslav Božić¹⁾, Hajnalka Peics²⁾, Aleksandar Milenković³⁾

¹⁾Educons University (Serbia)

²⁾University of Novi Sad (Serbia)

³⁾University of Kragujevac (Serbia)

Abstract. In this paper, the effectiveness of the use of multiple representations in teaching and learning functions is analyzed. The research was conducted in the frame of distance learning of calculus course at University of Novi Sad in 2021 (during COVID-19 crisis), in the dynamic software environment. The students' work on the analysis the properties of functions, by using the multiple representations of functions, is presented, and analyzed. Results of the study implies that dynamic software environment during distance learning of mathematics, encourage solving tasks with using different approach by students and helps them to create their own sophisticated knowledge about the functions and their properties.

Keywords: calculus; distance learning; dynamic software; functions; multiple representations

1. Introduction

It is known that the students have lot difficulties with the calculus contents within their mathematics courses (Borba&Confrey 1996); (Daher&Anabousy 2015); (Tall 1992); (Takač et al. 2015). Examining the properties of the functions and their multiple representations, algebraic and graphical, turned out to be one of most difficult students' tasks (Borba&Confrey 1996); (Ciltas&Tatar 2011); (Daher&Anabousy 2015); (Sierpinska et al. 2011); (Wilhelmi et al. 2007). Since the functions are used in a lot of the students' further courses and their professional work, it is very important for students to overcome the difficulties and the conflicts in understanding the function concept, emphasizing the absolute value functions. In the last few years, different dynamic packages, as *Mathematica*, *GeoGebra*, *Maple*, *Cinderella*, etc. are applied in teaching and learning the calculus contents. Almost all dynamic software packages are appropriate for examining the families of functions, algebraically represented as the functions with parameters. They enable simultaneous dynamic multiple representations of the families of functions and their properties, providing an easy change of variables (Anabousi et al. 2014); (Borba&Confrey 1996); (Ciltas&Tatar 2011); (Daher&Anabousy 2015).

The research was conducted with the students at University of Novi Sad. The research presented in this paper is the contribution to mathematics education during COVID-19, in particular to the function concept, in order to improve distance learning of the calculus in the future.

2. Theoretical background

2.1. Constructivism

Constructivism is a very important learning theory. Although this theory was present before, constructivism is practicing by numerous authors in the late twentieth and early twenty-first centuries (Bodner 1986); (Tobin & Tippins 1993); (Von Glaserfeld 1995). It is based on students' actively constructing their own knowledge. The students use their previous knowledge and experiences as the foundation for understanding new facts within their learning materials. As stated in (Von Glaserfeld 1995), "The constructivist model implies that knowledge is quality if it is applicable, that is, if it is useful in achieving some goals." For that reason, students will construct only the knowledge they need to solve a particular problem. According to (Iran-Nejad 1995); (Sjoberg 2010) and (Taber 2011), constructivism stimulates students' curiosity and their discussions, which should be of crucial importance in the learning process.

Although the role of the teacher is extremely important, the emphasis is more on what the students do. In constructivist learning approach, it is very important that the teacher carefully guides the students through the learning process. It is necessary to introduce them to a new area and then help them adapt and master new concepts and features in the most understandable way possible. The teacher should also help students eliminate difficulties that arise during learning and problem-solving process. The teachers should create such a learning environment that will help students acquire new skills and knowledge (Tobin & Tippins 1993). The teacher must develop a dialogue with the students during learning. The dialogue should be of high quality and supported by accurate information because students form their new knowledge based on it (Brooks & Brooks 1993); (Taber 2011). The key role of the teacher is to pose a problem that will be of interest to students and to feel the need to solve it (Von Glaserfeld 1995).

2.2. COVID-19 education

The educational process during the COVID-19 crisis was rapidly turned to the achievements of modern technology (Adedoyin & Soykan 2020); (Drijvers et al. 2021); (Fidalgo et al. 2020); (Mitrović et al. 2022); (Stevanović et al. 2021). Different audio and video conferencing, data and application sharing, shared whiteboard, virtual "hand raising" and so on, have been applied in each educational institution over 2020, 2021, and at the beginning of 2022. Namely, educational process turned forward future suddenly, varying from school to school, university to university.

According to (Psotka 2022), these days, almost after (and during) the COVID-19 crisis, it is important to work on research regarding different experiences of the teaching process, about teachers' and students' experiences, in order to contribute to the improvement of future education. All prepared teaching materials, students' and teachers' experiences need to be analyzed to improve the future teaching process.

2.3. Multiple representations

Representations in mathematics, as representations at all, are usually considered as a process of modeling concrete objects in the real world into abstract concepts (Hwang & Hu 2013). The observer puts two concepts one against the other, revealing and comparing the similarities and differences between them. In this way one of the observed concepts is represented (Font et al. 2007). The classification of representations was the subject of much research. Usually are considered two large groups of representations – internal representation, created in the mind of an individual, and external, created in his surroundings (Goldin & Shteingold 2001); (Nakahara 2008). A quality of the multiple representations is significantly improved by using modern technology. The use of technology to work within multiple representations and to link them has a great attention of the contemporary researchers (Rau et al. 2015); (Ozgun-Koca 2008); (Sever & Yerushalmy 2007).

The advantage of using computers in the formation of multiple representations is especially evident when connecting different representations. Namely, there are software packages that allow simultaneous display of two to three representations of the same object. More recently, dynamic software packages which enable forming the multiple representations are especially important. These software packages enable dynamically linking the multiple representations and creating the system of multiple representations, where a change in one representation causes simultaneous change in other representations of the same object (Hwang & Hu 2013). Dynamic software also enables forming of so-called virtual manipulative representation. This representation is an interactive visual representation of a dynamic object, which allows students to manipulate objects, i.e., to change one property of the object and at the same time to observe how the change of that property affects the other properties of the object. There are several dynamic software packages which give student opportunity to link different representations of mathematical objects. One of the more commonly used, because of its availability, simplicity, and performances, is *GeoGebra*. This software is used more often in teaching mathematics, from elementary school to university level. The application of *GeoGebra* was the topic of many researches in the last few years (Arzarello et al. 2012); (Doruk et al. 2013); (Takači et al. 2015).

2.4. Multiple representations of functions

Functions are mathematical concept that can be presented in many ways, i.e. by using different representations. One representation is usually not sufficient

to represent all the properties of a function. Therefore, multiple representations are usually applied to display the functions' properties (Božić & Takači 2021). Usually, the most importance is attached to the algebraic, graphic and tabular representation, but in recent times, more and more verbal representation is gaining importance. The use of the computers brought innovations in a representation of functions. Multiple representations of functions in a computer-based environment have been topic of many papers (Goldin & Shteingold 2001); (Borba & Confrey 1996); (Doorman, et al. 2012).

Multiple representations of functions are, in the research above, usually considered in order to improve students' achievements and to help them to overcome difficulties they have with functions' examining and graphing (Božić & Takači 2021); (Božić et al. 2022). Earlier researches have shown that students, when it comes to functions' examining, have the most difficulties in working with functions with variable parameters, namely families of functions, as well as with the transformations of functions (Borba & Confrey 1996); (Daher & Anabousy 2015); (Tall 1992).

The absolute value problems are usually related to the absolute value functions, equations and inequalities, and they are related to the absolute value functions. Most of the expressions, used in the absolute value equations and inequalities, can be observed as the transformations of the absolute value parent function, defined as $f(x)=|x|$ (Elis & Bryson 2011). Because of the mentioned, it is necessary for the students to deal with the absolute value functions and their properties, especially with the transformations of the functions. In (Elis & Bryson 2011) the authors point out that the application of the conceptual approach based on the combining different representations (including graphical and verbal) contributes to better achievements of the students, when it comes to the absolute value of the expression.

GeoGebra is one of the software packages which enables connecting of different representations and work within multiple representations of functions. It also enables forming the dynamic multiple representations of the functions, which are being formed by creating sliders, by which variable parameters are defined. The moving of the slider causes immediate changes of the parameter's value and, consequently, causes changes in algebraic and graphical representation, simultaneously.

3. Research question

The aim of this research is to examine the contribution of multiple representations, in the frame of distance learning during COVID-19 crisis, to the students' achievement in learning the properties of the absolute value functions. Due to the aim of the research, the research question is: How do students manage their learning by using multiple representations, in the dynamic soft-

ware environment, in order to successfully examine the properties of absolute value functions?

4. Methodology

4.1. General background

In this research the experimental approach was applied. During COVID-19 crisis the distance learning was enforced all over the world. In this situation the dynamic software package *GeoGebra* was used for examining the properties of related families of functions, in particular with absolute value functions. The benefits of students' work in a *GeoGebra* dynamic environment are examined and analyzed within a Calculus course at the University of Novi Sad, in 2021 year.

The distance learning was organized as a combination of the real-time lectures and sharing materials with the students. For the real-time lectures, *Microsoft Teams* was applied. The materials were shared by using *Microsoft Teams* and *GeoGebra materials*. The real-time lectures were organized in a way that the students actively participated and cooperated with the teacher and with the other students. All the lectures were recorded and shared with the students, so they were able to watch the videos later, according to their own needs. The students also got the tasks for their homework, which were assigned to them by using *Microsoft Teams*. In contrast to the learning process, the test was conducted in the classroom and the students were not able to use the computer during the test.

4.2. Participants

The research was conducted with 60 undergraduate students, during their calculus course, at the University of Novi Sad in 2021, during COVID-19 crisis.

4.3. Instruments and procedures

At the beginning of the research, the teacher renewed the students' knowledge of the elementary properties of functions, and as usual, special attention was paid to the functions of absolute value. Also, the students were introduced in using *GeoGebra* software.

The teachers did exercises with students concerning the multiple representations (algebraic, graphical, and verbal) of the functions $f(x) = x^2$ and $g(x) = x^2 + a$, $h(x) = |x^2 + a|$, $x \in R$, $a \in R$ shown in fig. 1. The functions f , g , h and the parameter a are inputted in *GeoGebra* dynamic software. Simultaneously, both their representations (algebraic and graphical) appeared, as is shown in fig. 1. Graphics view 2 is used, by the teacher, as an interactive whiteboard, showing each change of inputs. The parameter a , as the slider was moving, caused the change of the function g . The teacher conducted the analysis of the dependence of the properties of the function g , especially its zeros, on the parameter a , as it is shown in fig. 1.

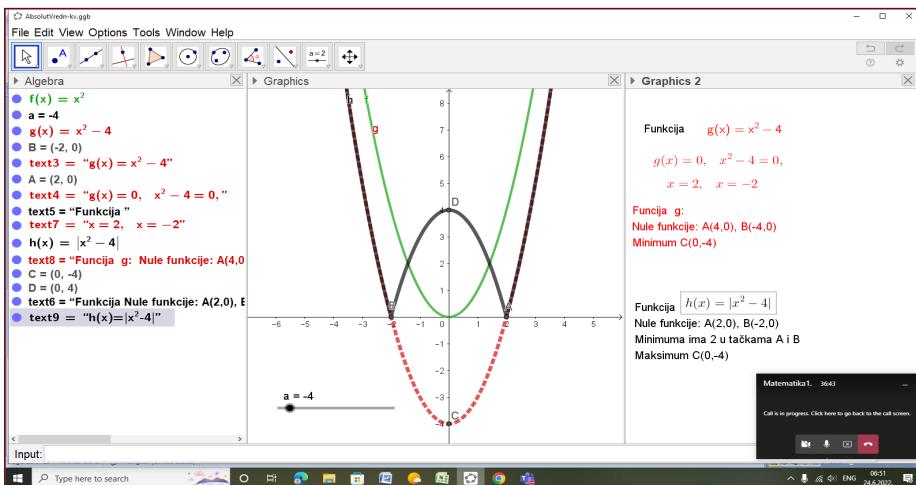


Figure 1. Teacher's explanation

During this lecture the students got the tasks (given in the Appendix A) concerning the properties of the absolute value functions. They had to finish and send them to the teacher individually within 180 minutes.

The first task was given in order to recall the students' knowledge about the elementary properties of functions. In the second and the third tasks, the students are required to examine the influence of the variable parameters on the properties of functions, with an accent on the absolute value. In all of these tasks the students had to analyze different functions, with zero, one, or two real parameters.

In the fourth task the students need to examine an absolute value function on two real parameters and its intersection with a line parallel to the x -axis. In fact, the students had to deal with three parameters. Therefore, this task was the most difficult one. Firstly, the students had to examine the influence of the connected parameters a and b to the properties of functions (at least four cases) and then to examine the influence of the parameter p on the number of solutions of the given equation depending on the function.

All these tasks are appropriate for examining functions under various representations (graphical, algebraic, and verbal). In fact, the graphical representations of considered functions are helpful for the required analysis.

In all given tasks, the students need to show their understanding of the dependence of the properties of absolute value functions on parameters by using their graphical representations, which can be obtained by using *GeoGebra*. Also, these graphical representations can be obtained without a computer. Since the functions were given with variable parameters, the students were suggested to use *GeoGebra*

package, because its dynamic properties enable simultaneous change of representations by changing parameters' values.

Examining the functions with variable parameters, within the dynamic multiple representations environment, enabled the students to perceive the influence of the parameters' values on the different functions' properties. Also, the students recalled their knowledge about the functions' properties by solving the first task, so they could analyze the differences between the absolute value functions' properties and the properties of the appropriate "non-absolute value" functions. This approach enabled the students to construct the knowledge about the absolute value functions and their properties, which points that learning environment, applied in this learning process, has the elements of the constructivist learning approach and the tasks are created in order to encourage constructivist learning of the students.

These tasks can be solved by using the algebraic representations of the given functions only, without the help of a computer, but the use of *GeoGebra* software is more convenient during the distance learning. After the exercises (and before the test), the teachers provided the explanations of the tasks for all students. Two weeks after their exercises, the students solved the test, at the University, without the computer. The test contained three tasks for a total time of 90 minutes. The tasks of the test are given in Appendix B.

In the first two tasks, the students had to use a graphical representation in order to analyze the absolute value functions $f(x) = |x^2 + a|$, $g(x) = ||x^2 + a| + b|$, $x, a, b \in R$. In fact, the purpose of these tasks was to test the students' knowledge of the influence of parameters on the properties of the absolute value functions, given by their graphs. Also, they had to examine the number of solutions of the absolute value equations. In the third task, the students had to deal with the absolute value equation given algebraically. The students' results of this test were analyzed.

5. Results

The tasks, envisaged for the students' exercises, are given in Appendix A. All students used *GeoGebra* software during their exercises and they were very satisfied with its help, because it enabled all tools necessary for examining functions. The absolute value functions were given without variable parameters only in the first task. In the other tasks all students used one, two or three sliders to change the values of parameters. For that purpose, they fixed a value of one (or two) parameters and, by changing the second (third) one, they got the same situation as in the first part of the task. As it was expected, all students worked on this task very successfully.

The teachers remarked that the students showed a great deal of variety in their work within *GeoGebra* dynamic environment. There were no two identical worksheets, although the tasks were the same. They used different resources of the *GeoGebra* package and got interesting algebraic and graphical representations of

the absolute value functions, which they used for examining the properties of these functions. This indicates that the students applied constructivist approach during their work, i.e., they constructed their knowledge about absolute value functions, but also the multiple representations of functions, in the way which was the most appropriate for them. Each student worked by his (her) own, analyzing different parameters and different functions, so they got correct results in different way.

In the following, the most interesting parts of works, from a mathematical point of view, are presented. For example, we present the picture of dynamic worksheet created by student S1 (fig. 2). He analyzed the similarities and the differences between the functions $f_1(x) = |x| + a$ and $f_4(x) = ||x| + a|$, by using “Graphics 1” and “Graphics 2” view, respectively. Both the graphical and algebraic representations of these functions are presented for $a=2$. Besides that, the student sent the verbal explanation where he correctly concluded that the observed functions are the same for the positive values of parameter a , but for negative values of this parameter they differ, and an example can be seen in fig. 2. It is expected that the examining and comparing the absolute value functions and their properties, such as the one conducted by student S1, increases the student’s abilities in the application of graphical representation for solving the absolute value equations.

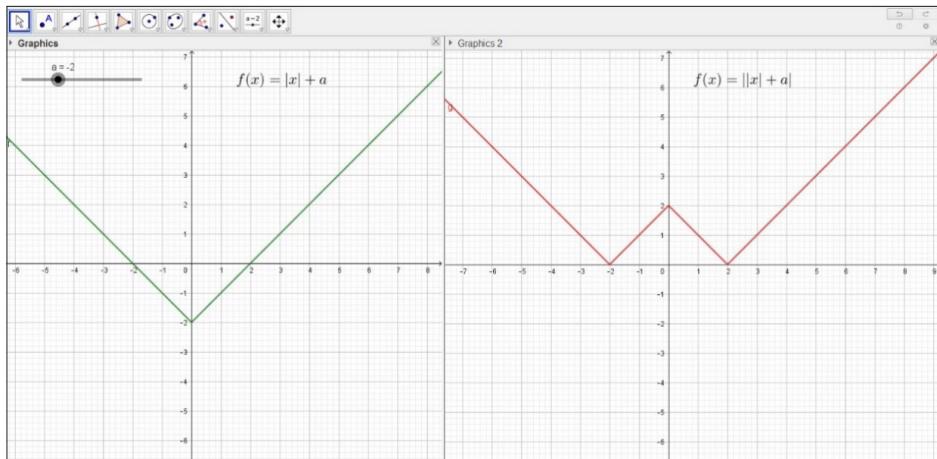


Figure 2. Student’s S1 work – comparing two functions in *GeoGebra*

The next examples of the successful applications of the *GeoGebra* properties are given in figures 3, 4, 5 and where the student’s S2 dynamic worksheets, used in solving the 4th task, are shown. This student analyzed four most important cases of the function $f(x) = ||(x + 2)^2 + a| + b|$ in dependence of the values of the real parameters a, b , and its connections. She started from the function $f_1(x) = (x + 2)^2$ and continues gradually with the graphs of functions

$$f_2(x) = (x+2)^2 + a, \quad f_3(x) = |(x+2)^2 + a|, \quad f_4(x) = ||(x+2)^2 + a| + b|$$

and she distinguished the following cases:

1. Case 1: The function f has only one minimum for $a>0, b>0$ or $b<0, |b|\leq a$. The graphs of f_1, f_2, f_3, f_4 for $a=3, b=-1$ are shown in fig. 3.
2. Case 2: The function f has one maximum and two minimums for $a>0$ and $|b|>a$ or $a\leq 0$ and $b>a$. The graphs of f_1, f_2, f_3, f_4 for $a=-3, b=1$ are shown in fig. 4.
3. Case 3: The function f has two maximums and three minimums for $a<0, b<0$ and $|b|\geq|a|$. The graphs of f_1, f_2, f_3, f_4 for $a=-1, b=-2$ are shown in fig. 5.
4. Case 4: The function f has three maximums and four minimums for $a<0, b<0$, and $|b|<|a|$. The graphs of f_1, f_2, f_3, f_4 for $a=-5, b=-2$ are shown in fig. 6.

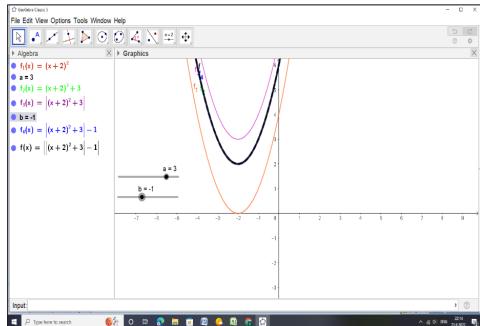


Figure 3. Student's S2 work – Case 1

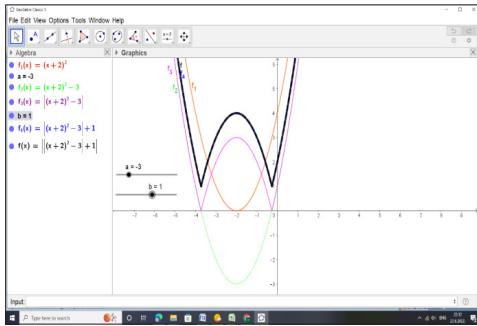


Figure 4. Student's S2 work – Case 2

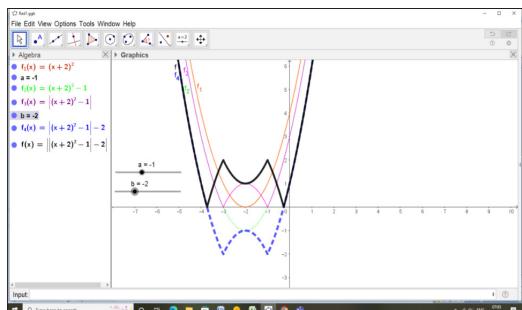


Figure 5. Student's S2 work – Case 3

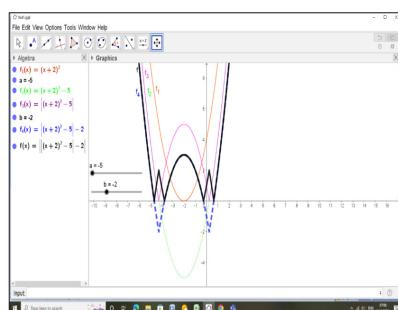


Figure 6. Student's S2 work – Case 4

In all these cases she paid special attention to the influence of the parameter p on the number of the solutions of the equation $f(x)=p$. Student S2 correctly applied options “intersection” in order to obtain the intersection points of the

graphic of the function $y=f(x)$ and the line $y=p$ whose x -coordinates represent the solutions of the equation $f(x)=p$. An example of students' S2 analysis of the equation $|(x+2)^2-1|-2|=p$ for $p=1$ is shown in fig. 7. Student S2 used the capabilities of the *GeoGebra* package in order to show only the graph of the function f (not the graphs of the other functions f_1, f_2, f_3, f_4) and to denote its points of intersection with the line $y=p$ by A, B, C, D , and E .

The teacher did not suggest the form of the finished students' tasks, so the students could prepare them by hand, by computer, or by combination thereof. The students prepared and submitted the final solutions in different ways. Among the final solutions 37 used a combination of hand and computer applications, while 23 were solved by computer only. The students who used their smart phones (24 students) submitted final solutions applying both methods, while 23 students working with desktop or laptop computers had only computer applications of solutions. Nobody had only handwritten solutions.

An example of the successful examining the properties of the function $f(x)=|x^2+a|$ (second task) depending on the parameter a , is given in fig. 8. Student S3 attached material as a combination of the electronic worksheet and handwritten document which is shown in fig. 8. We can remark that student S3 examined even more properties than it was required.

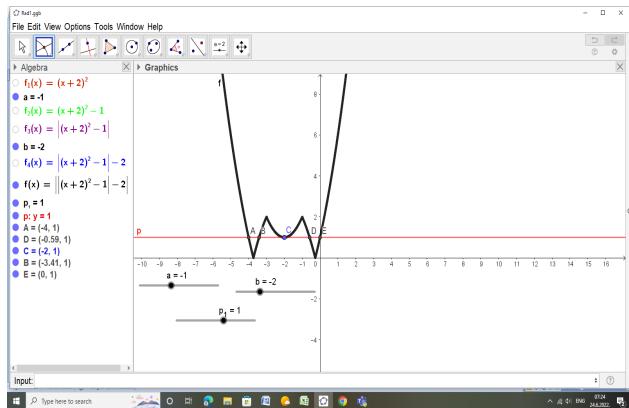


Figure 7. Student's S2 work

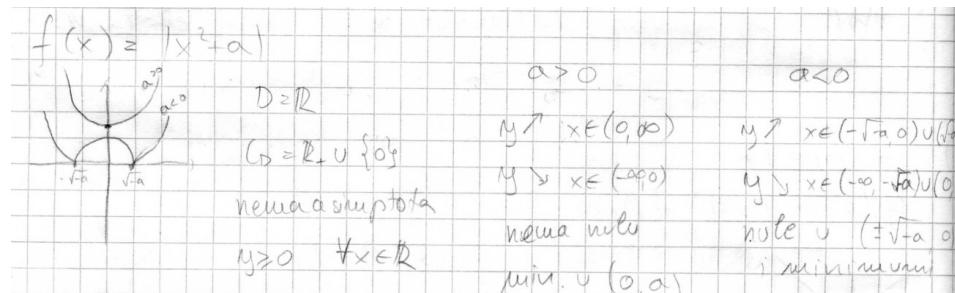


Figure 8. Student's S3 work – examining the properties of the function

GeoGebra environment enables examining functions by using simultaneous graphical and algebraic representations and geometric notions, such as the corresponding intersection of objects. For example, the students S1 and S2 used multiple representations, predominantly graphical. Most of the students took care about all required properties of the functions and their dependence on the parameters, but some of them omitted some properties. An overview of the number of students who correctly explained the influence of parameters on certain properties (interesting for this research) of the functions is given in Table 1.

In determining the zeroes and the extreme values of the given functions the students needed to denote the points, while in determining the signs of functions, the students needed to use the intervals. It is known that the students generally have problems with exact determining the corresponding intervals. Therefore, as it was expected, most of the students determined exactly the zeroes of the functions. The others just omitted some zeroes. There were no students who determined wrong coordinate for zero of function in all tasks. There were no students who did not determine exactly zeroes in the first task but determined correctly zeroes in the other tasks.

Table 1. Number of the students who correctly explained the influence of parameters on certain property of the function

Function's property

Task	Zeroes	Sign	Extremes
1.	52	36	48
2.	36	21	35
3.	18	15	26

Before the test, the teacher conducted a discussion with the students, using *Microsoft Teams*, analyzed their answers and suggested how to overcome the difficulties they encountered in their tasks.

6. Analysis of the test results

About two weeks after the exercises described in Section 4, the students' knowledge about the properties of absolute valued functions was tested. The test contained three tasks. Time for solving was 90 minutes. The students were not allowed to use computers during the test. The tasks of the test are given in Appendix B. The percentages of students who gave the correct results on the test for first two tasks are given in Table 2.

Table 2. Percentages of students with correct results on the test (for first two tasks)

Task	1A	1B	2A	2B	2C	2D
Correct a)	90%	78%	84%	68%	51%	36%
b)	64%	58%	64%	56%	43%	18%

By analyzing the results given in Table 2 we can notice that the students had the best results on the first task. The first part of the task was done better than the second part, because some of the students did not care about the values $p < 0$. There were 34 students (57%) who remarked that all functions were even, and it was useful for examining the number of solutions for given equations.

In the second task, referring Case 2A, there were 25 students (42%) who remarked that the solution is not unique and compared it with the Case 1A if one parameter equals zero. In solving the cases 2C and 2D the students worked successively considering the graphs of functions $y = x^2 + a$, $y = |x^2 + a|$, $y = |x^2 + a| + b$ and $y = ||x^2 + a| + b|$ for different real values of a and b . For example, the student's H1 work is shown in fig. 9. The student H1 drew the graphs of functions $y = |x^2 - 2|$, $y = |x^2 - 2| - 4$, $y = ||x^2 - 2| - 4|$ for $a = -2$, $b = -4$, with the same pencil but with different styles of lines. The student H2 drew all function for $a = -1$, $b = -4$, in different colors (fig. 10). As expected, the most difficult task was 2D; even 33% of students just omitted it.

It is interesting to note that all students had started to solve the third task. They explained that there were no arbitrary parameters and therefore they thought that it would not be difficult. The percentages of students who gave correct answers to the third task is presented in Table 3.

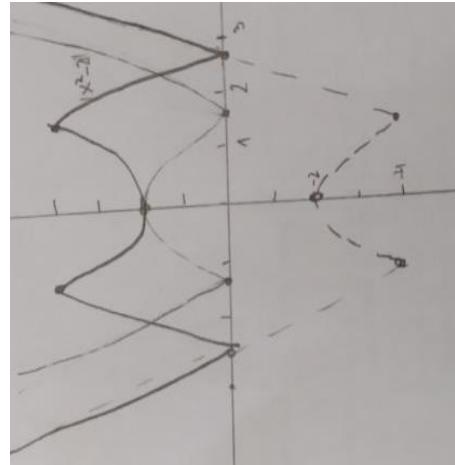


Figure 9. Student's H1 work – test, for $a = -2$, $b = -4$

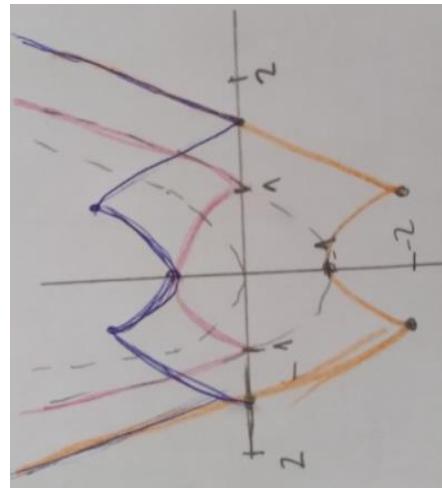


Figure 10.

Table 3. Percentages of students who gave correct answers to the third task

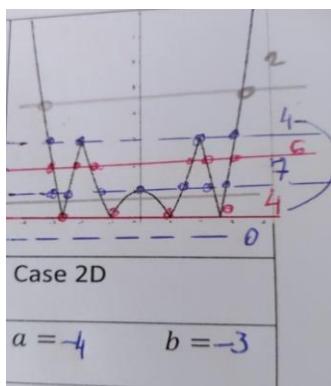
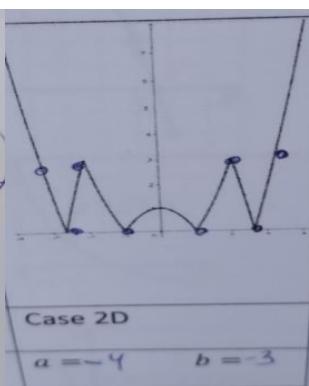
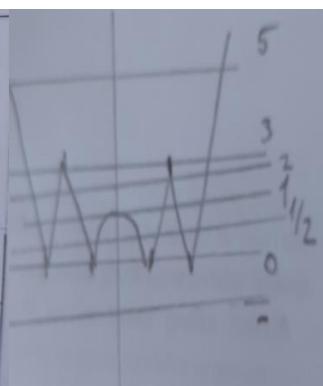
Task	3a	3b	3c
correct	36%	28%	13%

The students obtained the solutions of the third task working either only algebraically or by using graphical representation (combined with the algebraic representation). There were 39 students (65%) who combined the graphical and the algebraic representation in the third task (by using the method from the previous tasks), but still only 8 students got the correct solutions of this task.

Even 21 students (35%) were solving this task in algebraic representation, but none of them did correctly the task 3c). The task 3a) was the easiest, but only 10 students got its correct solution. Only 5 of those students, who correctly solved 3a), got correct solution of 3b). There were no students who had correct task 3b), but not correct 3a).

There were 18 students (33%) who, working the third task, recognized that the graph of the function $y=||x^2-4|-3|$, appearing on the left hand side of equations in the third task, is already drawn in case 2D, and used it for solving this task, i.e., for discussing the number of the solutions of equation $3c (||x^2-4|-3|=p)$.

The students' H3 and H4 works are presented in fig. 11 and fig. 12. They both used the graph in the case 2D. The task in fig. 11 is solved correctly. All interesting cases for the value p are analyzed and the numbers of solutions are given. fig. 12 shows the most common students' answers. These answers do not include all possible values of p . The student H4 did not even draw the graph of $y=p$. fig. 13 shows the work of student H5. He solved the task 3c) correctly, by drawing graph by himself. There were two students who consider the line $x=2$ instead of $y=2$ in the task 3b). This indicates that they were not able to use the graphical representations of the given equations.


Figure 11. Student's H3
work – test

Figure 12. Student's H4
work – test

Figure 13. Student's H5
work – test

7. Discussion

In this research, the influence and effectiveness of the application of multiple representations in the dynamic environment on students' achievements in examining the properties of the absolute value functions are analyzed in the frame of distance learning during COVID-19 crisis. The students had to use the multiple representations of the absolute value functions during their learning process and the test and to deal with the transformations of the absolute value functions. Such way of learning the absolute value functions is proved to be effective in learning the properties of these functions (Elis & Bryson 2011). In our research, we applied the multiple representations within the dynamic software environment. For providing the dynamic multiple representations of the absolute value functions, *GeoGebra* software package was applied, in accordance with (Stupel & Ben-Chaim 2014).

The students used *GeoGebra* dynamic software during the learning process in the frame of distance learning during COVID-19 crisis. The students' solutions of the tasks, written works and electronic materials, were reviewed by the teachers. There were difficulties related to the existence of more than one absolute value within the expression, like in the research (Ciltas & Tatar 2011), and to the existence of more than one variable parameter, in accordance with previous researches (Borba & Confrey 1996); (Daher & Anabousy 2015).

In order to check the efficiency of the use of multiple representation in learning the absolute value function, the students got the test with the appropriate tasks. The results of the test were analyzed. Besides the difficulties during the learning process, while solving the test the students had difficulties with choosing the appropriate method for solving the equations (algebraic only or combined – graphical and algebraic), which is in accordance with the research (Stupel 2012) and (Stupel & Ben-Chaim 2014).

The analysis of the students' work on exercises and students' test showed great way of variety in their finished tasks. This indicates that they applied constructivist way of learning, according to (Bodner 1986), (Tobin & Tippins 1993) and (Von Glaserfeld 1995) – the students constructed their knowledge about the absolute value functions in accordance with their own learning needs, which resulted with numerous different approaches in solving the tasks. The teacher created a learning environment (Microsoft Teams, with *GeoGebra* package) and encouraged discussion with the students about their difficulties during working on their exercises, in order to help students to acquire new skills and knowledge (Brooks & Brooks 1993); (Iran-Nejad 1995); (Sjoberg 2010); (Taber 2011). The students also spoke highly of the use of *GeoGebra* during classes where they practiced tasks and deepened their mathematical knowledge. They especially liked being able to relate the algebraic notations of the functions to their graphs and see how changing a parameter in the algebraic notation of the function affected its graph.

The research presented in the paper contains students' experiences during COVID-19 in learning functions and, according to (Psotka 2022), this can represent the contribution to the improvement of future education.

8. Conclusions

In our research we analyze the contribution of multiple dynamic representations to the improvement of the students' learning process concerning the function concept in the frame of distance learning during COVID-19 crisis. The students organized their learning in different ways, depending on their previous knowledge and their learning skills and interests. This was expressed in the variety of ways the students used to solve the tasks using *GeoGebra* software, which is discussed in the analysis of students' work.

From the above discussion it can be concluded that the dynamic software environment during distance learning encourages the students to apply different ways and in solving tasks, i.e. to adjust the learning methods to their needs, and helps students to create their own sophisticated knowledge about the functions and their properties.

There exist some limitations of this research, of course. Namely, the tasks chosen for the analysis were appropriate for graphical solving and it is shown that the dynamic software contributes to the students' achievements in this field. But, in the future research, other types of the absolute value problems should be considered in the environment of the modern technology.

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Appendix A

The tasks solved by the students during their work on examining the absolute value functions

1. Examine the properties (zeros, sign, extreme values) of the functions $|f(x)|$ and $f(|x|)$ where the functions f are given in the table below:

$f(x) = -5x + 2$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = x^{-1}$
$f(x) = 2^x$	$f(x) = 1/3^x$	$f(x) = \log x$	$f(x) = \sin x$

2. Examine the properties (zeros, sign, extreme values) of the given functions in dependence of real parameter a :

$f_1(x) = x + a$	$f_2(x) = x + a $	$f_3(x) = a x $
$f_4(x) = x + a $		$f_5(x) = x^2 + a $

3. Examine the properties (zeros, sign, extreme values) of the given functions in dependence of real parameter a and b

$g_1(x) = a x + b$	$g_2(x) = x + a + b$	$g_3(x) = x + a + b $
$g_4(x) = x^2 + a + b$	$g_5(x) = x^2 + a + b $	

4. Examine the influence of the real parameters a , b , p on the properties (zeros, extreme values) of function $f(x) = |(x-2)^2 + a| + b|$. Determine the number of solutions of the equation $f(x) = p$.

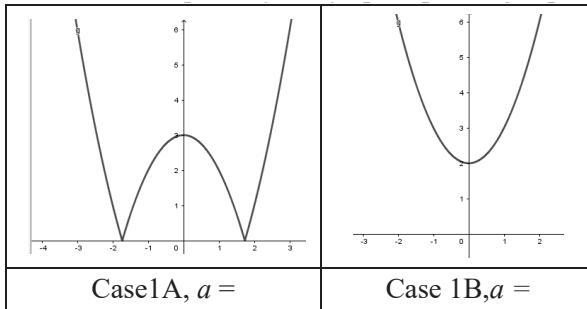
Appendix B

The tasks for the test.

1. The function $f(x) = |x^2 + a| + b$ is given with real parameter a .

a) Determine the value of parameter a so, that the graphs below correspond to the function f and explain the answers for the cases 1A and 1B.

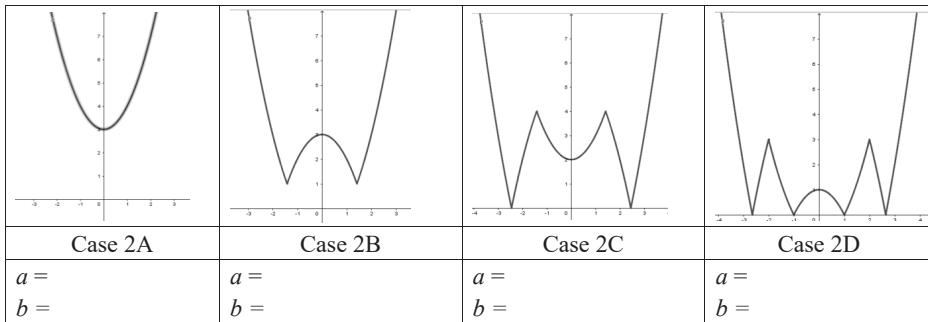
b) Discuss the number of solutions of equation $|x^2 + a| + p$, depending on p , for the cases below:



2. The function $g(x) = |x^2 + a| + b$ is given with the real parameters a and b .

a) Determine the values of parameters a and b , so that the graphs below correspond to the function f and explain the answers.

b) Discuss the number solutions of equation $|x^2 + a| + b = p$ depending on real parameter p , for the cases below:



3. Determine the number of solutions of the equations

a) $|x^2 - 4| - 3 = 0$, b) $|x^2 - 4| - 3 = 2$, c) $|x^2 - 4| - 3 = p$, $p \in \mathbb{R}$.

✉ Dr. Radoslav Božić, Assist. Prof.

ORCID iD: 0000-0003-1128-8823

Educons University

Faculty of Pedagogy

Novi Sad, Serbia

E-mail: radoslav.bozic@gmail.com

✉ Prof. Dr. Hajnalka Peics

ORCID iD: 0000-0002-1182-6647

University of Novi Sad

Faculty of Civil Engineering

Subotica, Serbia

E-mail: peics@gf.uns.ac.rs

✉ Dr. Aleksandar Milenković, Assist. Prof.

ORCID iD: 0000-0001-6699-8772

University of Kragujevac

Faculty of Science

Kragujevac, Serbia

E-mail: aleksandar.milenkovic@pmf.kg.ac.rs