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Educational Issues
Въпроси на преподаването

MIDLINES OF QUADRILATERAL

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Abstract. In this section we deal with an interesting fact which will imply *ABCD* some well known theorems. Namely we prove that in a convex quadrilateral whose diagonals meet at O, the midline of the triangle ΔBOC parallel to *BC*, the midline of the triangle ΔAOD parallel to *AD*, and the midline of the quadrilateral *ABCD* connecting the sides *AB* and *CD* meet at one point, or are parallel.

Keywords: Ceva; Menelaus; Stewartes; cevian; concurrency; collinearity; Fermat; Torricelli

1. INTRODUCTION

Observing the history of geometry, we can see that one of the most attractive problems was the concurrency of three lines, and collinearity of three points. Two astonishing results connecting those two problems are Ceva and Menelaus theorems, well known to most of the mathematicians, especially competitive ones. Here we present a theorem in quadrilateral, implying both of those theorems and also Van Aubels theorem which is well conected to those two. We also present interesting collinearity and concurrency in quadrilateral, and another equaivalent form of Ceva's theorem which sometimes yields the desired result faster than its standard form. The main point is that these results are derived using only similarity and midlines of a triangles which are well known to the primary school students as well as the beginners of high school students. Concurency of three midlines in quadrilateral, as we will see, yields various results and identities. Once the main theorem is proved we can use the identities obtained by itself to prove Stewart's theorem which yields the lenghts of various cevians, such as a median, an altitude, an angle bisector etc. Depending on what kind of quadrilateral we choose, we can get any point inside the triangle, even extremal points, such as Fermat-Torricelli, centroid and similar ones.

Assumption. Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at O. Let P,Q,R,S,M,N be the midpoints of the segments AO,BO,CO,DO,AB,CD respectively. Let MN, meet QR,AC,BD at the points V,T,U respectively. Let also VO meet PQ,QM and AB at the points K,L and Z respectively.

2. LIST OF IDENTITIES 1

(1) $NR = \frac{DO}{2}$

Proof:

Since N is the midpoint of CD,and R is the midpoint of CO, thus NR is a midline of triangle ΔCOD, hence $NR = \frac{DO}{2}$

(2) $SN = \frac{CO}{2}$

Proof:

Since N is the midpoint of CD, and S is the midpoint of DO, thus NS is a midline of triangle ΔCOD, hence $SN = \frac{CO}{2}$

(3) $MQ = \frac{AO}{2}$

Proof:

Since M is the midpoint of AB, and Q is the midpoint of BO, thus MQ is a midline of triangle ΔBOA, hence $MQ = \frac{AO}{2}$

(4) $SQ = \frac{BD}{2}$

Proof:

Since S is the midpoint of DO, and Q is the midpoint of BO, thus

$$SQ = SO + QO = \frac{DO}{2} + \frac{BO}{2} = \frac{DO + BO}{2} = \frac{BD}{2}$$

(5) $UQ = \frac{BD \cdot AO}{2AC}$

Proof:

Since SN is the midline of ΔCOD we have $SN||CO \Rightarrow SN||AC$.
Since MQ is the midline of ΔBOA we have $MQ||AO \Rightarrow MQ||AC$.
Hence we have $SN||MQ$, so $\Delta SNU \sim \Delta MUQ$, so we have

$$\frac{US}{UQ} = \frac{NS}{MQ} \Rightarrow \frac{US}{UQ} + 1 = \frac{NS}{MQ} + 1 \Rightarrow \frac{SQ}{UQ} = \frac{NS + MQ}{MQ} \Rightarrow$$

$$UQ = \frac{SQ \cdot MQ}{SN + MQ} =^{(4),(3)} \frac{\frac{BD}{2} \cdot \frac{AO}{2}}{\frac{CO}{2} + \frac{AO}{2}} = \frac{BD \cdot AO}{2AC}$$

(6) $UO = \dfrac{AO \cdot DO - CO \cdot BO}{2AC}$

Proof: If UQ<QO then this case is impossible. So let it be UQ>QO

$$UO = UQ - QO =^{(5)} \frac{BD \cdot AO}{2AC} - \frac{BO}{2} = \frac{BD \cdot AO - AC \cdot BO}{2AC} =$$
$$= \frac{(BO + DO)AO - (AO + CO)BO}{2AC} = \frac{AO \cdot DO - BO \cdot CO}{2AC}$$

(7) $US = \dfrac{BD \cdot CO}{2AC}$

$$US = QS - UQ =^{(4),(5)} \frac{BD}{2} - \frac{BD \cdot AO}{2AC} = \frac{BD}{2AC}(AC - AO) = \frac{BD \cdot CO}{2AC}$$

(8) $PR = \dfrac{AC}{2}$

Proof:
Since P is the midpoint of AO, and R is the midpoint of CO, thus

$$PR = PO + RO = \frac{AO}{2} + \frac{CO}{2} = \frac{AO + CO}{2} = \frac{AC}{2}$$

(9) $TR = \dfrac{AC \cdot DO}{2BD}$

Proof:

Since RN is the midline of ΔCOD we have $RN||DO \Rightarrow RN||BD$

Since MP is the midline of ΔBOA we have $MP||BO \Rightarrow MP||BD$

Hence we have $RN||MP$, so $\Delta RTN \sim \Delta PMT$, so we have

$$\frac{PT}{RT} = \frac{PM}{NR} \Rightarrow \frac{PT}{RT} + 1 = \frac{PM}{NR} + 1 \Rightarrow \frac{PR}{RT} = \frac{PM + NR}{NR} \Rightarrow$$

$$RT = \frac{PR \cdot NR}{PM + NR} =^{(8),(1)} \frac{\frac{AC}{2} \cdot \frac{DO}{2}}{\frac{BO}{2} + \frac{DO}{2}} = \frac{AC \cdot DO}{2BD}$$

(10) $PT = \dfrac{AC \cdot BO}{2BD}$

$$PT = PR - RT =^{(9)} \frac{AC}{2} - \frac{AC \cdot DO}{2BD} \Rightarrow \frac{AC}{2}(BD - DO) \Rightarrow$$

$$PT = \frac{AC \cdot BO}{2BD}$$

$$(11) \quad TO = \frac{AO \cdot DO - BO \cdot CO}{2BD}$$

$$TO = PO - PT =^{(10)} \frac{AO}{2} - \frac{AC \cdot BO}{2BD} = \frac{BD \cdot AO - AC \cdot BO}{2\,BD} =$$

$$= \frac{(BO + DO) \cdot AO - (AO + CO) \cdot BO}{2\,BD} = \frac{AO \cdot DO - BO \cdot CO}{2BD}$$

$$(12) \quad PM = \frac{BO}{2}$$

Proof:

Since M is the midpoint of AB, and P is the midpoint of AO, thus AO is a midline of triangle ΔBOA, hence $PM = \frac{BO}{2}PM = \frac{BO}{2}$

$$(13) \quad PS = \frac{AD}{2}$$

Proof:

Since P is the midpoint of AO and S is the midpoint of DO, then PS is a midline of triangle ΔDOA, so

$$PS = \frac{AD}{2}$$

$$(14) \quad NU = \frac{MN \cdot CO}{AC}$$

Proof:

Since $SN || MQ$, we have $\Delta SUN \sim \Delta MUQ$, so

$$\frac{MU}{NU} = \frac{MQ}{SN} \Rightarrow \frac{MU}{NU} + 1 = \frac{MQ}{SN} + 1 \Rightarrow \frac{MN}{UN} = \frac{MQ + SN}{SN} \Rightarrow$$

$$UN = \frac{MN \cdot SN}{MQ + SN} =^{(2),(3)} \frac{MN \cdot \frac{CO}{2}}{\frac{AO}{2} + \frac{CO}{2}} = \frac{MN \cdot CO}{AC}$$

(15) $MT = \frac{MN \cdot BO}{BD}$

Proof:
Since $PM||RN$, we have $\Delta PTM \sim \Delta RTN$

$$\frac{NT}{MT} = \frac{NR}{PM} \Rightarrow \frac{NT}{MT} + 1 = \frac{NR}{PM} + 1 \Rightarrow \frac{MN}{MT} = \frac{NR + PM}{PM} \Rightarrow$$

$$MT = \frac{MN \cdot PM}{NR + PM} =^{(1),(12)} \frac{MN \cdot \frac{BO}{2}}{\frac{DO}{2} + \frac{BO}{2}} = \frac{MN \cdot BO}{BD}$$

(16) $RQ = \frac{BC}{2}$

Proof:
Since R is the midpoint of CO, and Q is the midpoint of BO, thus RQ is a midline of triangle ΔBOC, hence $RQ = \frac{BC}{2}$

(17) $PQ = \frac{AB}{2}$

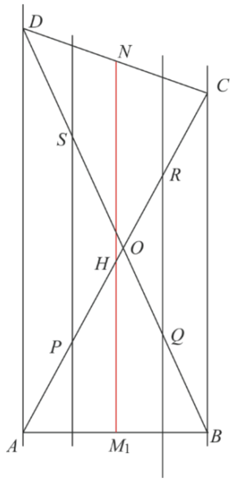
Proof:
Since P is the midpoint of AO, and Q is the midpoint of BO, thus PQ is a midline of a triangle ΔBOA, hence $PQ = \frac{AB}{2}$

3. MAIN THEOREM

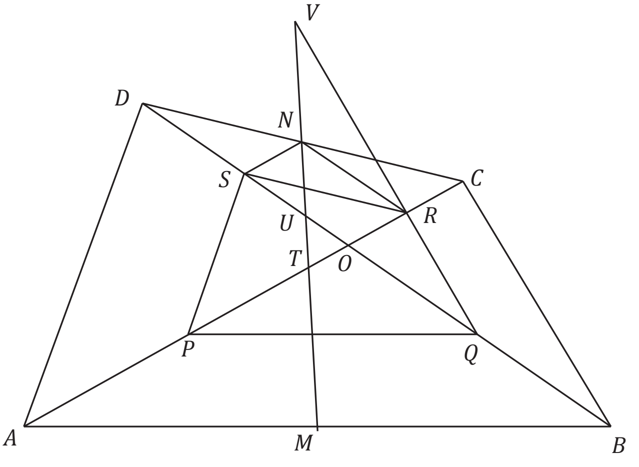
Theorem 1. Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at O. Let P, Q, R, S, M, N be the midpoints of the segments AO, BO, CO, DO, AB, CD respectively. Then the lines PS, QR and MN meet at one point, or are parallel.

Proof:
Let $AD||BC$ and let the line parallel to BC meets AC and AB at the points H and M_1 respectively. Since N is the midpoint of CD and $NH||AD$, then HN is a midline of triangle ΔACD, hence the point H is a midpoint of AC. Since H is the midpoint of AC and $HM_1||BC$ then HM_1 is a midline of trian-

gle ΔABC, hence the point M_1 is the midpoint of AB. So we have $M_1 \equiv M$, so the lines AD, BC, MN are parallel. Since PS is the midline of triangle ΔADO, then $PS||AD$. Since QR is a midline of triangle ΔBOC, then $QR||BC$. So we get the lines PS, QR and MN are parallel. Let AD be non-parallel to BC.



Pic.1



Pic. 2

We will consider the case when $A_{\Delta AOB} > A_{\Delta COD}$. The other case is similar. Let MN meet QR, AC, BD at the points V, T, U respectively.

$$\frac{VM}{VN} = \frac{\frac{MN \cdot AO \cdot BO}{AO \cdot BO - CO \cdot DO}}{\frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO}} = \frac{AO \cdot BO}{CO \cdot DO} \Rightarrow \frac{VM}{VN} - 1 = \frac{AO \cdot BO}{CO \cdot DO} - 1 \Rightarrow$$

$$\frac{MN}{VN} = \frac{AO \cdot BO - CO \cdot DO}{CO \cdot DO} \Rightarrow$$

$$VN = \frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO}$$

Suppose on the other side that PS and MN meet at W. Then similarly

$$WN = \frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO} \Rightarrow V \equiv W$$

In the former calculations we will denote this point by V.

4. LIST OF IDENTITIES 2

$$(18)\quad VS = \frac{AD \cdot BD \cdot CO}{2(AO \cdot BO - CO \cdot DO)}$$

Proof:
Since $PM||SU$, we have $\Delta VUS \sim \Delta VMP$, so

$$\frac{VP}{VS} = \frac{PM}{US} \Rightarrow \frac{VP}{VS} - 1 = \frac{PM}{US} - 1 \Rightarrow \frac{PS}{VS} = \frac{PM - US}{US} \Rightarrow$$

$$VS = \frac{PS \cdot US}{PM - US} =^{(7),(12),(13)} \frac{\frac{AD}{2} \cdot \frac{BD \cdot CO}{2AC}}{\frac{BO}{2} - \frac{BD \cdot CO}{2AC}} = \frac{AD \cdot BD \cdot CO}{2(AC \cdot BO - BD \cdot CO)} =$$

$$= \frac{AD \cdot BD \cdot CO}{2(AO + CO)BO - 2(BO + DO)CO} = \frac{AD \cdot BD \cdot CO}{2(AO \cdot BO - CO \cdot DO}$$

$$(19)\quad VR = \frac{AC \cdot BC \cdot DO}{2(AO \cdot BO - CO \cdot DO)}$$

Proof:
Since $QM||RT$, we have $\Delta VRT \sim \Delta VMQ$, so

$$\frac{VQ}{VR} = \frac{QM}{RT} \Rightarrow \frac{VQ}{VR} - 1 = \frac{QM}{RT} - 1 \Rightarrow \frac{RQ}{VR} = \frac{QM - RT}{RT} \Rightarrow$$

$$VR = \frac{RT \cdot RQ}{QM - RT} =^{(9),(3),(16)} \frac{\frac{AC \cdot DO}{2BD} \cdot \frac{BC}{2}}{\frac{AO}{2} - \frac{AC \cdot DO}{2BD}} = \frac{AC \cdot BC \cdot DO}{2(BD \cdot AO - AC \cdot DO)} =$$

$$= \frac{AC \cdot BC \cdot DO}{2(BO + DO)AO - (AO + CO)DO} = \frac{AC \cdot BC \cdot DO}{2(AO \cdot BO - CO \cdot DO)}$$

$$(20)\quad VQ = \frac{BD \cdot BC \cdot AO}{2(AO \cdot BO - CO \cdot DO)}$$

Proof:

$$VQ = VR + RQ =^{(19),(16)} \frac{AC \cdot BC \cdot DO}{2(AO \cdot BO - CO \cdot DO)} + \frac{BC}{2} =$$

$$= \frac{BC}{2(AO \cdot BO - CO \cdot DO)}(AC \cdot DO + AO \cdot BO - CO \cdot DO) =$$

$$\frac{BC}{2(AO\cdot BO-CO\cdot DO)}(AO\cdot DO+CO\cdot DO+AO\cdot BO-CO\cdot DO)=$$

$$=\frac{BC}{2(AO\cdot BO-CO\cdot DO)}(AO\cdot DO+AO\cdot BO)=\frac{BC\cdot BD\cdot AO}{2(AO\cdot BO-CO\cdot DO)}$$

$$(21)\quad VP=\frac{AD\cdot AC\cdot BO}{2(AO\cdot BO-CO\cdot DO)}$$

Proof:

$$VP=PS+SV=^{(13),(18)}\frac{AD}{2}+\frac{AD\cdot BD\cdot CO}{2(AO\cdot BO-CO\cdot DO)}=$$

$$=\frac{AD}{2(AO\cdot BO-CO\cdot DO)}(AO\cdot BO-CO\cdot DO+BO\cdot CO+CO\cdot DO)$$

$$=\frac{AC\cdot AD\cdot BO}{2(AO\cdot BO-CO\cdot DO)}$$

$$(22)\quad QL=\frac{BD\cdot AO\cdot CO}{2AC\cdot DO}$$

Proof:
Since $RO||QL$ we have $\Delta VRO\sim\Delta VLQ$, so we have

$$\frac{QL}{RO}=\frac{VQ}{VR}\Rightarrow QL=^{(19),(20)}\frac{RO\cdot VQ}{VR}=\frac{\frac{CO}{2}\cdot\frac{BD\cdot AO}{2(AO\cdot BO-CO\cdot DO)}}{\frac{AC\cdot BC\cdot DO}{2(AO\cdot BO-CO\cdot DO)}}=$$

$$=\frac{BD\cdot BC\cdot AO\cdot CO}{2AC\cdot BC\cdot DO}=\frac{BD\cdot AO\cdot CO}{2AC\cdot DO}$$

$$(23)\quad KP=\frac{AC\cdot AB\cdot DO}{2(BD\cdot CO+AC\cdot DO)}$$

Proof:
Since $PO||QL$ we have $\Delta POK\sim\Delta LKQ$, so we have

$$\frac{QK}{KP}=\frac{QL}{PO}\Rightarrow\frac{QK}{KP}+1=\frac{QL}{PO}+1\Rightarrow\frac{PQ}{KP}=\frac{QL+PO}{PO}\Rightarrow$$

$$KP = \frac{PO \cdot PQ}{QL + PO} =^{(22),(17)} \frac{\frac{AO}{2} \cdot \frac{AB}{2}}{\frac{BD \cdot AO \cdot CO}{2AC \cdot DO} + \frac{AO}{2}} = \frac{AC \cdot AB \cdot DO}{2(BD \cdot CO + AC \cdot DO)}$$

(24) $QK = \frac{BD \cdot AB \cdot CO}{2(BD \cdot CO + AC \cdot DO)}$

Proof:
Since $PO||QL$ we have $\Delta POK \sim \Delta LKQ$, so we have

$$QK = PQ - KP =^{(17),(23)} \frac{AB}{2} - \frac{AC \cdot AB \cdot DO}{2(BD \cdot CO + AC \cdot DO)} =$$

$$= \frac{AB(BD \cdot CO + AC \cdot DO - AC \cdot DO)}{2(BD \cdot CO + AC \cdot DO)} = \frac{BD \cdot AB \cdot CO}{2(BD \cdot CO + AC \cdot DO)}$$

(25) $VN = \frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO}$

Proof:
Since NR is the midline of triangle ΔCOD, thus $NR || BD \Rightarrow^{(1)} NR || UQ$. So we have similarity $\Delta VUQ \sim \Delta VNR$, thus

$$\frac{VU}{VN} = \frac{UQ}{NR} \Rightarrow \frac{VU}{VN} - 1 = \frac{UQ}{NR} - 1 \Rightarrow \frac{UN}{VN} = \frac{UQ - NR}{NR} =^{(1),(5)}$$

$$= \frac{\frac{BD \cdot AO}{2AC} - \frac{DO}{2}}{\frac{DO}{2}} = \frac{BD \cdot AO - AC \cdot DO}{AC \cdot DO} = \frac{(BO + DO)AO - (AO + CO)DO}{AC \cdot DO} =$$

$$= \frac{AO \cdot BO - CO \cdot DO}{AC \cdot DO} \Rightarrow$$

$$VN = \frac{UN \cdot AC \cdot DO}{AO \cdot BO - CO \cdot DO} = \frac{\frac{MN \cdot CO}{AC} \cdot AC \cdot DO}{AO \cdot BO - CO \cdot DO} = \frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO}$$

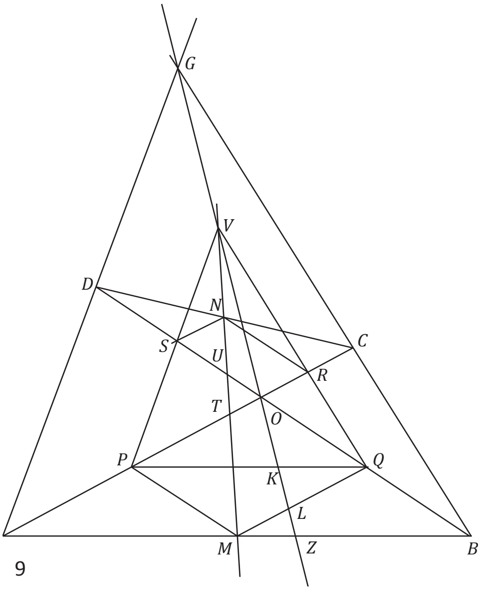
(26) $VM = \frac{MN \cdot AO \cdot BO}{AO \cdot BO - CO \cdot DO}$

Proof:

$$VM = VN + MN \overset{(25)}{=} \frac{MN \cdot CO \cdot DO}{AO \cdot BO - CO \cdot DO} + MN = \frac{MN \cdot AO \cdot BO}{AO \cdot BO - CO \cdot DO}$$

5. SUBMAIN THEOREM

Theorem 2.Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at O. Let P, Q, R, S, M, N be the midpoints of the segments AO, BO, CO, DO, AB, CD respectively. Let AD not parallel to BC and let the lines PS, QR and MN meet at a point V. Let the lines AD and BC meeet at G. Then the points G, V, O are collinear.



Pic. 3

Proof:

Let the line OV meet BC at G_1. Since QR is a midline of triangle ΔBOC, then $QR||BC$ so we have $QV||BG_1$. Since Q is the midpoint of OB and $QV||BG_1$ then QV is a midline of triangle ΔBOG_1, hence V is midpoint of OG_1.

If we suppose the line OV meets AD at the point G_2, similarly we obtain that the point V is midpoint of OG_2.

So we have $G_1 \equiv G_2 \equiv G$.

After we have proved Theorem 2, we can see that the triangles ΔABG and ΔPVQ are homothetic with respect to O and the absolute value of the coefficient of homotethy is 2.

We will now show that the consequences of these theorems are some well known theorems. The converses of those theorems (if exist) are trivial, so we will not deal with them

6. CONSEQUENCES

Corolary 1 (Menelaus theorem)

Let the line p meets segments AB, BC and the extension of the segment AC at the points C_1, A_1 and B_1 respectively.Then

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = 1$$

Proof:

Consider the line OG and the triangle ΔADB. It suffices to prove

$$\frac{AZ}{ZB} \cdot \frac{BO}{OD} \cdot \frac{DG}{AG} = 1$$

Using identities (23) and (24); (18) and (21) we have

$$\frac{2PK}{2KQ} \cdot \frac{BO}{OD} \cdot \frac{2VS}{2PV} =$$

$$= \frac{\frac{AC \cdot AB \cdot DO}{2(BD \cdot CO + AC \cdot DO)}}{\frac{BD \cdot AB \cdot CO}{2(BD \cdot CO + AC \cdot DO)}} \cdot \frac{BO}{OD} \cdot \frac{\frac{AD \cdot BD \cdot CO}{2(AO \cdot BO - CO \cdot DO)}}{\frac{AD \cdot AC \cdot BO}{2(AO \cdot BO - CO \cdot DO)}} =$$

$$= \frac{AC \cdot DO}{BD \cdot CO} \cdot \frac{BO}{OD} \cdot \frac{BD \cdot CO}{AC \cdot BO} = 1,$$

which completes the proof.

Corolary 2 (Cevas theorem)

Let A_1, B_1, C_1 be the points on the segments BC, CA, AB respectively such that AA_1, BB_1 and CC_1 meet at one point. Then

$$\frac{AB_1}{B_1C}\cdot\frac{CA_1}{A_1B}\cdot\frac{BC_1}{C_1A}=1$$

Proof:
Consider the point O and the triangle ΔGAB. It suffices to prove

$$\frac{AZ}{ZB}\cdot\frac{BC}{CG}\cdot\frac{GD}{DA}=1.$$

Using identities (23) and (24); (19) , (18) we have

$$\frac{2PK}{2KQ}\cdot\frac{BC}{2VR}\cdot\frac{2VS}{DA}=$$

$$=\frac{\frac{AC\cdot AB\cdot DO}{2(BD\cdot CO+AC\cdot DO)}}{\frac{BD\cdot AB\cdot CO}{2(BD\cdot CO+AC\cdot DO)}}\cdot\frac{BC}{\frac{AC\cdot BC\cdot DO}{AO\cdot BO-CO\cdot DO}}\cdot\frac{\frac{AD\cdot BD\cdot CO}{AO\cdot BO-CO\cdot DO}}{DA}=1,$$

which completes the proof.

Corolary 3 (Van Aubels theorem)
Let A_1, B_1, C_1 be the points on the segments BC, CA, AB respectively such that AA_1, BB_1 and CC_1 meet at one point O.Then

$$\frac{AO}{OA_1}=\frac{AB_1}{B_1C}+\frac{AC_1}{C_1B}$$

Proof:
Consider the point O and the triangle ΔABG. It suffices to prove

$$\frac{AO}{OC}=\frac{AZ}{ZB}+\frac{AD}{DG}$$

Using identities(23), (24), (18) we have

$$\frac{AZ}{ZB}+\frac{AD}{DG}=\frac{2PK}{2KQ}+\frac{AD}{2VS}=\frac{\frac{AC\cdot AB\cdot DO}{BD\cdot CO+AC\cdot DO}}{\frac{BD\cdot AB\cdot CO}{BD\cdot CO+AC\cdot CO}}+\frac{AD}{\frac{AD\cdot BD\cdot CO}{AO\cdot BO-CO\cdot DO}}=$$

$$= \frac{AC \cdot DO}{BD \cdot CO} + \frac{AO \cdot BO - CO \cdot DO}{BD \cdot CO} = \frac{AO \cdot DO + AO \cdot BO}{BD \cdot CO} = \frac{BD \cdot AO}{BD \cdot CO} = \frac{AO}{OC}$$

which completes the proof.

Theorem 3 (Reformulation of Cevas theorem).

Let C_1 and B_1 be the points on the sides AB and CD of a triangle ΔABC such that CC_1 and BB_1 meet at O. Let A_1 be the point on the side BC. Line AA_1 passes through the point O if and only if

$$\frac{BA_1}{A_1C} \cdot \frac{CC_1}{BB_1} \cdot \frac{B_1O}{C_1O} = 1$$

Proof:

Using our triangle it suffices to prove

$$\frac{AZ}{BZ} \cdot \frac{BD}{AC} \cdot \frac{CO}{DO} = 1$$

Since QK is a midline of triangle ΔBOZ, and KP is a midline of triangle ΔAZO, then we have

$$\frac{AZ}{BZ} \cdot \frac{BD}{AC} \cdot \frac{CO}{DO} = \frac{PK}{QK} \cdot \frac{BD}{AC} \cdot \frac{CO}{DO} =$$

$$\frac{\frac{AC \cdot AB \cdot DO}{2(BD \cdot CO + AC \cdot DO}}{\frac{BD \cdot AB \cdot CO}{2(BD \cdot CO + AC \cdot DO)}} \cdot \frac{BD}{AC} \cdot \frac{CO}{DO} = \frac{AC \cdot DO}{BD \cdot CO} \cdot \frac{BD}{AC} \cdot \frac{CO}{DO} = 1$$

which completes the proof.

Corolary 4 (Stewart's theorem)

Let ΔABC be a given triangle, and let D be a point on the segment BC. Then

$$AD^2 = \frac{AC^2 \cdot BD + BC^2 \cdot AD - BC \cdot AD \cdot BD}{BC}$$

Proof:

Consider the point C and the triangle ΔABG. t suffices to prove

$$AC^2 = \frac{AB^2 \cdot CG + AG^2 \cdot BC - BG \cdot BC \cdot GC}{BG}$$

Using the identities we have

$$\frac{AB^2 \cdot CG + AG^2 \cdot BC - BG \cdot BC \cdot GC}{BG} = \frac{AB^2 \cdot 2VR + 4VP^2 \cdot BC - 2VQ \cdot BC \cdot 2VR}{2QV} =$$

$$= AB^2 \cdot \frac{AC \cdot DO}{BD \cdot AO} + \frac{AD \cdot AC \cdot BO}{AO \cdot BO - CO \cdot DO} \cdot \frac{AD \cdot AC \cdot BO}{BD \cdot AO} - \frac{BC^2 \cdot AC \cdot DO}{AO \cdot BO - CO \cdot DO} =$$

$$= AC^2 [\frac{AB^2 \cdot DO}{BD \cdot AC \cdot AO} + \frac{AD^2 \cdot BO^2}{BD \cdot AO(AO \cdot BO - CO \cdot DO)} - \frac{BC^2 \cdot DO}{AC(AO \cdot BO - CO \cdot DO)}] =$$

$$= AC^2 \cdot \frac{AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + AD^2 \cdot BO^2 \cdot AC - BC^2 \cdot BD \cdot AO \cdot DO}{BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)} =$$

$$= AC^2 \cdot \frac{AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + AD^2 \cdot BO^2 \cdot AC - BC^2 \cdot BD \cdot AO \cdot DO}{BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)}$$

Now we will use some known identities for the quadrilateral:

$$\frac{AD^2 - AO^2 - DO^2}{AO \cdot DO} = \frac{AO^2 + BO^2 - AB^2}{AO \cdot BO} \Rightarrow$$

$$AD^2 = \frac{BO \cdot DO \cdot BD + AO^2 \cdot BD - AB^2 \cdot DO}{BO}$$

and

$$\frac{BC^2 - BO^2 - CO^2}{BO \cdot CO} = \frac{AO^2 + BO^2 - AB^2}{AO \cdot BO} \Rightarrow$$

$$BC^2 = \frac{AO \cdot CO \cdot AC + BO^2 \cdot AC - AB^2 \cdot CO}{AO}$$

Using those identities we have

$$AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + AD^2 \cdot BO^2 \cdot AC - BC^2 \cdot BD \cdot AO \cdot DO =$$

$$AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + \frac{BO \cdot DO \cdot BD + AO^2 \cdot BD - AB^2 \cdot DO}{BO} \cdot BO^2 \cdot AC$$

$$- \frac{AO \cdot CO \cdot AC + BO^2 \cdot AC - AB^2 \cdot CO}{AO} \cdot BD \cdot AO \cdot DO =$$

$$AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + BO^2 \cdot DO \cdot BD \cdot AC + AO^2 \cdot BO \cdot BD \cdot AC -$$

$$AB^2 \cdot DO \cdot BO \cdot AC - AO \cdot CO \cdot AC \cdot BD \cdot DO - BO^2 \cdot AC \cdot BD \cdot DO + AB^2 \cdot CO \cdot BD \cdot DO$$

$$= AB^2[BO \cdot DO(AO - AC) + CO \cdot DO(BD - DO)] + BD \cdot AC \cdot AO(AO \cdot BO - CO \cdot DO) =$$

$$AB^2 \cdot (-BO \cdot DO \cdot CO + CO \cdot DO \cdot BO) + BD \cdot AC \cdot AO(AO \cdot BO - CO \cdot DO) =$$

$$BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)$$

So we have

$$AC^2 \cdot \frac{AB^2 \cdot AO \cdot BO \cdot DO - AB^2 \cdot CO \cdot DO^2 + AD^2 \cdot BO^2 \cdot AC - BC^2 \cdot BD \cdot AO \cdot DO}{BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)} =$$

$$= AC^2 \cdot \frac{BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)}{BD \cdot AC \cdot AO \cdot (AO \cdot BO - CO \cdot DO)} = AC^2,$$

Which completes the proof.

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