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METHOD FOR SOLVING SYSTEM OF LINEAR EQUATIONS MXN, M=1, 2, 3; BASED ON ITS GEOMETRICAL INTERPRETATIONS

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Abstract. In this paper, by using the geometrical classification of the types of linear equations or systems of two equations, we determine a method for solving systems of linear equations. The analysis specifically refers to systems of linear equations that reduce to a quadratic system of the second or third order in which all determinants of the system are zero. The methodological significance of the method is also discussed.

Keywords: system of linear equations; slassifications of systems; geometrical classification

1. Introduction

We consider a system of m linear equations with n unknowns ($m \times n$) over the set of the real numbers. Let A be the matrix and $\overline{A}$ be the augmented matrix of the system $m \times n$. Let $r(A)$ and $r(\overline{A})$ be the ranks of A and $\overline{A}$, respectively. Let for $m \leq n+1$; $D_{i_1 i_2 \ldots i_m}$, where $i_1, i_2, \ldots, i_m$ is a permutation without repetition of a class m of the set $I_{n+1} = \{1, 2, \ldots, n+1\}$, be a minor of $\overline{A}$ whose columns are matching with $i_1, i_2, \ldots, i_m$ column of $\overline{A}$, respectively.

The generalized Cramer's rule (Akhtyamov et al. 2017), divides the systems $m \times n$, into the following three classes (the statement for $m \leq n$ is given in (Misajleski et al. 2021)):

1. There exists $D_{i_1 i_2 \ldots i_m} \neq 0$ for some $i_1, i_2, \ldots, i_m \in I_n$. Then the system has a unique solution for $m = n$, and infinite solutions for $m < n$, expressed through $n-m$ parameters. The solution can be expressed explicitly by using the Cramer's formulas.

2. $D_{i_1 i_2 \ldots i_m}$ does not exist or $D_{i_1 i_2 \ldots i_m} = 0$ for every $i_1, i_2, \ldots, i_m \in I_n$; and

$D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} \neq 0$ for some $i_1^0, i_2^0, \dots, i_{m-1}^0 \in I_n$. In this case the system does not have solution.

3. $D_{i_1 i_2 \dots i_m}$ does not exist or $D_{i_1 i_2 \dots i_m} = 0$ for every $i_1, i_2, \dots, i_m \in I_{n+1}$. Then the system does not have a solution or has infinitely many solutions, expressed through $n - r(A)$ parameters, where $n - r(A) \in \{n - m + 1, \dots, n-1, n\}$.

These classes will be called ***Cramer classes***.

Definition 1.1. *Zero* linear equation is the equation in which all coefficients are zero. *Contradictory* linear equation is the equation in which all coefficients of the system (before unknowns) are zero, while the constant term is different from zero. *Regular* linear equation is an equation in which at least one of the coefficients of the system is different than zero. Two linear equations are *contradictory to each other* if they are regular and do not have a common solution. We say that $m \geq 2$ equations are equivalent if they have same solutions. *One linear equation is in a general position* if it is a regular. m *linear equations are in a general position* if they consist an $m-1$ linear equations in a general position and the m-th equation is not equivalent or contradictory to any linear combination of the $m-1$-th equations.

The next theorem is proven in (Misajleski et al. 2021) for $m \leq n$. We extend the statement for arbitrary positive integer n.

Theorem 1.2. For the system of m linear equations with n unknowns:

1. There exists $D_{i_1^0 i_2^0 \dots i_m^0} \neq 0$, for some $i_1^0, i_2^0, \dots, i_m^0 \in I_n$ if and only if the m-th equations are in a general position.

2. $D_{i_1 i_2 \dots i_m}$ does not exist or $D_{i_1 i_2 \dots i_m} = 0$ for every $i_1, i_2, \dots, i_m \in I_n$ and there exists $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} \neq 0$ for some $i_1^0, i_2^0, \dots, i_{m-1}^0 \in I_n$ if and only if there exist a $m-1$ equations in a general position, and the m-th equation is a contradictory to some linear combination of the other equations.

3. $D_{i_1 i_2 \dots i_m}$ does not exist or $D_{i_1 i_2 \dots i_m} = 0$ for every $i_1, i_2, \dots, i_m \in I_{n+1}$ if and only if there exist $m-1$ equations in a general position, and the m-th equation is an equivalent to some linear combination of the other equations, or do not exist $m-1$ equations in a general position.

Proof. For $m \leq n$ all determinants exist and the theorem is proven in Misajleski et al. (2021). If $D_{i_1 i_2 \dots i_m}$, $i_1, i_2, \dots, i_m \in I_n$; does not exist and there exists $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)}$ then $n = m-1$. $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} \neq 0$ means that: $D_{i_1^0 i_2^0 \dots i_{m-1}^0} \neq 0$, i.e. there exist $m-1$ equations in a general position, and m-th equation is a contradictory to some linear combination of the other equations (see the determinants in (Misajleski et al. 2021, pp. 42)). So, if $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} \neq 0$ holds 2) and if $D_{i_1^0 i_2^0 \dots i_{m-1}^0 (n+1)} = 0$ holds 3). If $D_{i_1 i_2 \dots i_m}$ does not exist for every $i_1, i_2, \dots, i_m \in I_{n+1}$ it follows that $n \leq m-2$ and then do not exist $m-1$ equations in a general position.■

The *geometrical classes* given in Misajleski et al. (2021), will be explained more studiously and counted in another way using the linear combination of the equations. They are the basis of the method of solving systems of linear equations presented in section two.

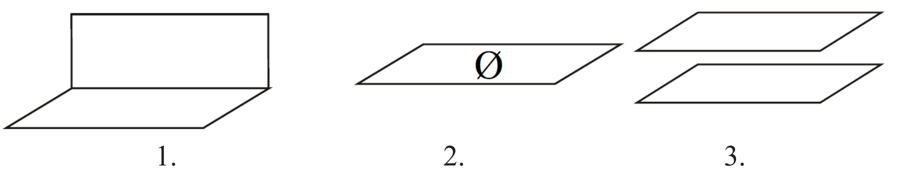
The set of linear equations with n unknowns is divided into 3 classes: regular, contradictory and zeros equations. We will call these classes geometric classes for $m=1$, because of their geometric interpretations. The classes are non-empty for each positive integer n. Since for $n=3$ regular linear equation is interpreted with the plane, the zero equation with the space, and the contradictory equation with the empty set, we write the classes with the following symbols:

Figure 1. Classification of the linear equations

Let the system of 2 linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \end{cases}$ be given.

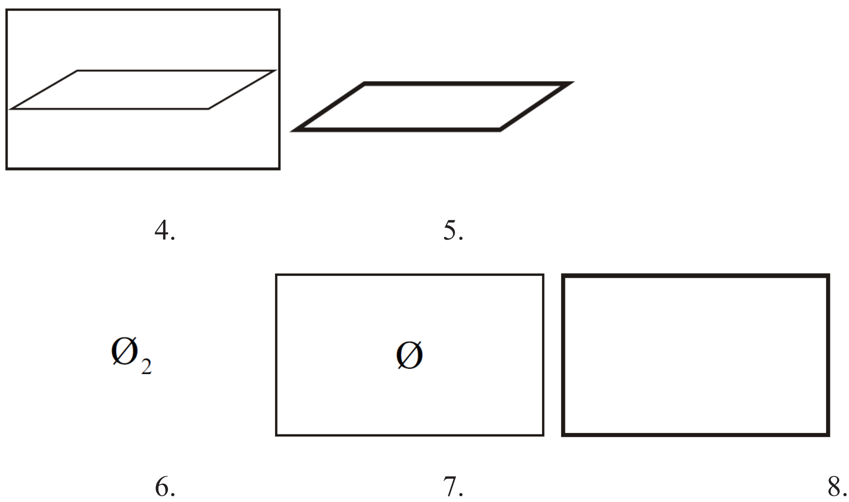
Let one of the equations, denoted by Σ, be regular. The linear combination $\lambda\Sigma$ of Σ is the zero equation for $\lambda=0$ or equivalent equation with Σ for $\lambda \neq 0$. If the other equation Π is regular and is not equivalent or contradictory to the linear combination of Σ, then the equations Σ and Π are in general position (class 1). If Π is contradictory to $\lambda\Sigma$, then for $\lambda=0$, Π is a contradictory equation (class 2) or, for $\lambda \neq 0$ the equations Σ and Π are contradictory to each other (class 3). If Π is equivalent to $\lambda\Sigma$, then Π is zero for $\lambda=0$ (class 4) or for $\lambda \neq 0$ is equivalent to Σ (class 5). If none of the equations is regular, then it is possible the two equations to be contradictory (class 6), to be contradictory and zero (class 7)) and both to be zero (class 8). In this way, for $m=2$ we formed 8 geometric classes.

For $n=3$, two regular equations that are: in general position, equivalent, or contradictory to each other, are interpreted by two planes that intersect, coincide, or are parallel, respectively. Therefore, we can write the geometric classes symbolically as in figure 2. When $n=1$ the first geometric class is empty. Therefore for $n=1$ there exist 7 nonempty geometric classes.



1. 2. 3.

1. $D_{i_0 j_0} \neq 0$, $i_0, j_0 \in I_n$; 2-3. $D_{ij}=0$, $i,j \in I_n$; and there exists $D_{i_0(n+1)} \neq 0$, $i_0 \in I_n$.



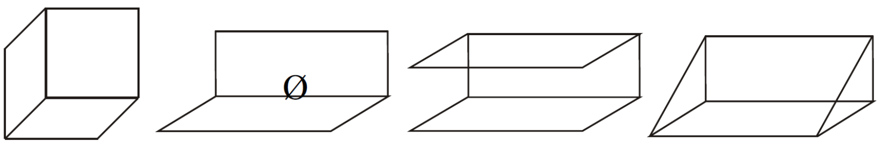
4 – 8. $D_{ij} = 0$, $i, j \in I_{n+1}$.

Figure 2. Classification of the systems of two linear equations

Let the system of 3 linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n = b_3 \end{cases}$ be given.

It is possible that the three equations are in general position (class 1). If there exists two equations Σ and Π in general position, then their linear combination $\lambda\Sigma + \mu\Pi$ is a zero equation when $\lambda = \mu = 0$; equivalent equation of Σ or Π when exactly one of the coefficients is zero and an equation containing the intersection points of Σ and Π when both coefficients are nonzero. Therefore if the third equation is contradictory to the linear combination, then it is contradictory (class 2), contradictory to one of the two equations (class 3) or contradictory to the equation that contain the intersection points of Σ and Π (class 4); while if it is equivalent to the linear combination, then it is zero (class 5), equivalent to one of the equations (class 6), or an equation that contain the intersection points of Σ and Π (class 7). Let the system do not contain two equations in general position. If the three equations are regular then they can be pairwise mutually contradictory (class 8), two equations to be contradictory to each other and equivalent to the third equation (class 9), and all three to be equivalent (class 10). If the system contains two regular equations, then they can be contradictory to each other and the third one to be contradictory (class 11) or zero (class 12); or the two equations to be equivalent and the third contradictory (class 13) or zero (class 14). If the system has one regular equation then the other two are contradictory (class 15), contradictory and zero (class 16) or both are zero (class 17). If the system does not contain regular equations then the three equations are contradictory (class 18), two contradictory and one zero (class 19), two are zero and one contradictory (class 20) or the three equations are zero (class 21).

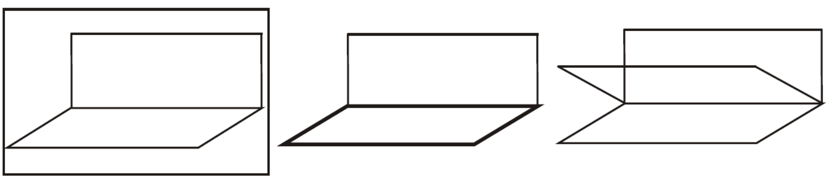
For $n=3$, three equations in general position are interpreted by three planes that intersect at a point. If a plane is a linear combination of two planes with two non-zero coefficients, then the three planes belong to the same sheaf of planes. The other interpretations of the equations are explained earlier. Therefore, the geometric classes for $m=3$ can be written symbolically as follows:



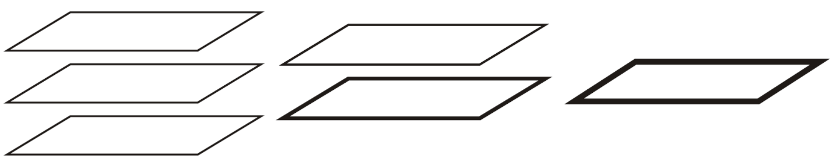
1. 2. 3. 4.

1. There exists $D_{i_0 j_0 k_0} \neq 0$, $i_0, j_0, k_0 \in I_n$;

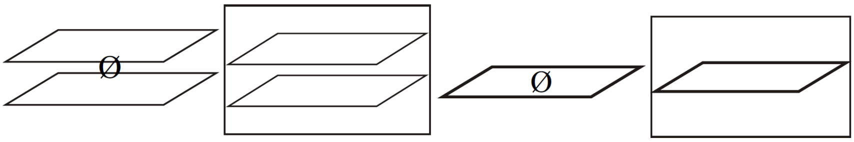
2 – 4. $D_{ijk} = 0$, $i, j, k \in I_n$ and there exists $D_{i_0(n+1)} \neq 0$, $i_0 \in I_n$.



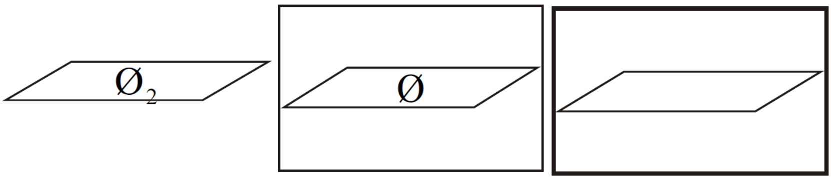
5. 6. 7.



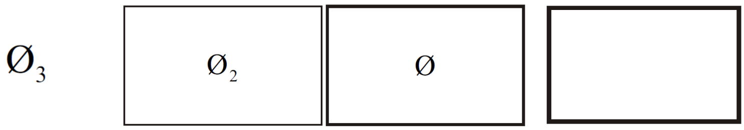
8. 9. 10.



11. 12. 13. 14.



15. 16. 17.



18. 19. 20. 21.

$5-21.\ D_{ijk}=0$ for every $i,j,k\in I_{n+1}$.

Figure 3. Classification of the systems of three linear equations

For $n\geq 3$ all 21-th class are nonempty. For $n=2$ there are no three equations in general position, i.e. the first class is empty, so there are 20 nonempty geometric classes. For $n=1$ there are no two equations in general position, so the first 7 classes are empty and there 14 nonempty geometric classes. The geometric classes are subclasses of the Cramer classes (Misajleski et al. 2021).

We will describe one more classification. A system $m\times n$ can be classified into a $2m+1$ class in relation to $r(A)$ and $r(\bar{A})$. Geometric classes are subclasses of this classification as well.

If $m=3$ for the 1-th geometric class $r(A)=r(\bar{A})=3$, for the 2-th to 4-th class $r(A)=2$ and $r(\bar{A})=3$, 5-th to 7-th $r(A)=r(\bar{A})=2$, 8-th to 14-th $r(A)=1$ and $r(\bar{A})=2$, 15-th to 17-th $r(A)=r(\bar{A})=1$, for 18-th to 20-th $r(A)=0$ and $r(\bar{A})=1$, for 21-th $r(A)=r(\bar{A})=0$.

If $m=2$ for the 1-th geometric class $r(A)=r(\bar{A})=2$, 2-th to 3-th $r(A)=1$ and $r(\bar{A})=2$, 4-th to 5-th $r(A)=r(\bar{A})=1$, 6-th to 7-th $r(A)=0$ and $r(\bar{A})=1$ and for 8-th $r(A)=r(\bar{A})=0$.

If $m=1$ for 1-th class $r(A)=r(\bar{A})=1$, 2-th $r(A)=0$ and $r(\bar{A})=1$ and for 3-th $r(A)=r(\bar{A})=0$.

2. Solving systems of linear equations $m\times n$, $m\in\{1,2,3\}$ based on the geometrical interpretations on equations

2.1. Solving linear equations

The null equations have infinitely many solutions expressed in terms of n

parameters (the set of solutions is arbitrary n-tuple of real numbers). Contradictory equations have no solution. Regular equations have a unique solution for $n=1$ and infinite solutions for $n>1$ expressed in terms of $n-1$ parameter. The solutions can easily be written explicitly, such that the unknown whose coefficient is nonzero will be expressed through the others that will be taken for parameters.

Example 2.1.1. The equation $0x+0y+0z+0u+0v=0$ is a zero equation and the set of solutions is arbitrary 5-tuple of real numbers.

Example 2.1.2. The equation $0x+0y+0z=1$ is contradictory, and has no a solution.

Example 2.1.3. The equation $2x=3$ is a regular equation with 1 unknown and has a unique solution $x=\frac{3}{2}$.

The equation $2x+0y=3$ is a regular equation with 2 unknowns and has infinite solutions expressed in terms of $2-1=1$ parameter: $\left(\frac{3}{2},t\right)$, t real number.

2.2. Discussion on the equations in system of two linear equations

The regular equations:

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1$ и $a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2$.

1. Are equivalent if and only if there exists a nonzero number k such that: $a_{21}=ka_{11}, a_{22}=ka_{12},\ldots,a_{2n}=ka_{1n}, b_2=kb_1$.

2. Are contradictory to each other if and only if there exists a nonzero number k such that: $a_{21}=ka_{11}, a_{22}=ka_{12},\ldots,a_{2n}=ka_{1n}$ and $b_2\neq kb_1$.

3. The equations are in a general position if there exists no nonzero number k such that: $a_{21}=ka_{11}, a_{22}=ka_{12},\ldots,a_{2n}=ka_{1n}$.

The number a is proportional to b, notation $a{:}\,b$, if there exists a nonzero number k such that $a=kb$. So, $0{:}\,0$ and $a{:}\,b$, $a,b\neq 0$. Otherwise $a\not{:}\,0$ when $a\neq 0$; and $0\not{:}\,b$ when $b\neq 0$.

We say the numbers $a_1,a_2,\ldots,a_n$ are respectively proportional to $b_1,b_2,\ldots,b_n$, notation $a_1{:}\,b_1=a_2{:}\,b_2=\ldots=a_n{:}\,b_n$, if the proportions $a_i{:}\,b_i$, $i=1,2,\ldots,n$; exist and have a common coefficient of proportion. In that case the numbers a_i, b_i either both are zero or both are non-zero, for each $i\in I_n$. Then $a_{i_1},a_{i_2},\ldots,a_{i_k}$, $i_1<i_2<\ldots<i_k$; are all nonzero elements of the sequence $a_1,a_2,\ldots,a_n$ if and only if $b_{i_1},b_{i_2},\ldots,b_{i_k}$ are all nonzero elements of the sequence $b_1,b_2,\ldots,b_n$ and $\frac{a_{i_1}}{b_{i_1}}=\frac{a_{i_2}}{a_{i_2}}=\ldots=\frac{a_{i_k}}{a_{i_k}}$. Thus:

1. Two linear equations are equivalent if and only if the coefficients are proportional, i.e. $a_{21}:a_{11}=a_{22}:a_{12}=...=a_{2n}:a_{1n}=b_2:b_1$,

2. Two linear equations are contradictory to each other if and only if they are regular and the coefficients before the unknowns are proportional, i.e. $a_{21}:a_{11}=a_{22}:a_{12}=...=a_{2n}:a_{1n}$, but all coefficients are not proportional, i.e. does not exist $b_2:b_1$ or exist $a_{2i},a_{1i},b_2:b_1\neq 0$ and $\frac{a_{2i}}{a_{1i}}\neq\frac{b_2}{b_1}$ for some $i\in I_n$.

3. Two linear equations are in general position if and only if the coefficients before the unknowns are not proportional, i.e. does not exist $a_{2i}:a_{1i}$ for some $i\in I_n$ or there exist a four nonzero coefficients $a_{2i}:a_{1i},a_{2j},a_{1j}$ such that $\frac{a_{2i}}{a_{1i}}\neq\frac{a_{2j}}{a_{1j}}$.

2.3. Solving system of two linear equations

Theorem 2.3.1. Consider the system: $\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2 \end{cases}$.

1. If the equations are in a general position, the system has an unique solution for $n=2$ or infinitely many solutions for $n>2$, expressed through $n-2$ parameters.
2. If the system contains a contradictory or two contradictory to each other equations, then the system has no solution.
3. If one of the equations is regular and the other is equivalent to the first equation or is zero, then the solutions of the system coincide with the solutions of the regular equation, so the system has a unique solution for $n=1$ or infinitely many solutions for $n>1$, expressed through $n-1$ parameter.
4. If the two equations in the system are zero, then the set of solutions is expressed through n parameters.

Proof. The accuracy of the theorem follows directly from the geometrical classification of systems of 2 linear equations.

If two equations are in general position, then $n\geq 2$ and the system is of geometric class 1. The system is solving by one of the known methods. In the substitution method, one of the unknowns whose coefficient is non-zero is expressing through the remaining $n-1$ unknowns, and is replacing in the other equation, where another unknown whose coefficient is non-zero is expressing through the remaining $n-2$ unknowns. Then it is replacing in the first equation and the first variable is expressing through the $n-2$ unknowns that will be taken as parameters. The system can also be solved with Cramer‘s formulas. Since $D_{i_0j_0}\neq 0$ for some $i_0,j_0\in I_n$, $i_0<j_0$; the system reduces to 2×2, such that all unknowns except x_{i_0} and x_{j_0} (total $n-2$), will be taken as parameters. Then $x_{i_0}=D'_{i_0(n+1)}/D_{i_0j_0}$ and $x_{j_0}=D'_{j_0(n+1)}/D_{i_0j_0}$, where $D_{i_0j_0}$ is the main, while $D'_{i_0(n+1)}$ and $D'_{(n+1)j_0}$ the

secondary determinants in the newly formed system.

If the system contains contradictory or contradictory to each other equations then it is of class 2, 3, 6 or 7, and has no solution. If one of the equations is regular and the other is equivalent to the first or zero, then the system is of class 4 or 5, and its solutions coincide with the solutions of the regular equation. If both equations are zero, then the system is of class 8 and every ordered n-tuple is a solution. So, the theorem covers all systems of two equations.

The theorem enables, without solving, to know when the system has no solution, when the solution is unique and in cases in which the system has infinitely many solutions, with how many parameters the solutions are described. The zeros and contradictory equations in the system are obvious. The mutual position of two regular equations is also obvious, where the relations of the pairs of non-zero coefficients are analyzed. To make the analysis clearer, we will consider the following examples.

Example 2.3.2. The equations in the system $\begin{cases} 3x-y=2 \\ 6x-2y=4 \end{cases}$ are equivalent ($\frac{3}{6}=\frac{-1}{-2}=\frac{2}{4}$). It follows that the system has infinitely many solutions expressed through $2-1=1$ parameter, which are obtained by rejecting of any of the equations. If the second equation is discarded, the system reduces to the equation $3x-y=2$. If $x=t$, $t\in\mathbb{R}$, then $y=3t-2$. It follows that the solutions are: $(t,3t-2)$, for t real number.

Example 2.3.3. The equations in the system $\begin{cases} 2x+2y+4z=7 \\ x+y+2z=3 \end{cases}$ are contradictory to each other ($\frac{2}{1}=\frac{2}{1}=\frac{4}{2}\neq\frac{7}{3}$). It follows that the system has no solution.

Example 2.3.4. The equations in the system $\begin{cases} 2x+0y+4z+u=7 \\ x+0y+2z+u=3 \end{cases}$ are in general position ($\frac{2}{1}\neq\frac{1}{1}$), it follows that the system has infinitely many solutions expressed by $4-2=2$ parameters. The solutions $(4-2t,k,t,-1)$, for t,k real numbers, are obtained by some of the standard methods.

The equations in the system $\begin{cases} 2x - 0y = 2 \\ x + y = 3 \end{cases}$ are in general position ($0 \not{:} 1$). It follows that the system has a unique solution. From the first equation $x = 1$, and from the second $y = 2$, i.e. the solution is $(1, 2)$.

Example 2.3.5. In the system $\begin{cases} 0x + 0y + 0z = 3 \\ x + y + 2z = 7 \end{cases}$ the first equation is contradictory, so the system has no solution.

Example 2.3.6. In the system $\begin{cases} 0x + 0y = 0 \\ 0x + 0y = 0 \end{cases}$ both equations are zero, so the set of solutions is is arbitrary ordered couple of real numbers.

2.4. Solving a system of three linear equations

The mutual relationship of three regular equations of the geometric classes 1, 4 and 7 is not always obvious. But those classes are in different Cramer classes. Therefore, the following theorem first determines the Cramer classes using a determinants of third order, and then determines the number of solutions using the geometric classes.

Theorem 2.4.1. Let a system $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \end{cases}$ be given. Then:

1. If there exists a $D_{i_0 j_0 k_0} \neq 0$, $i_0, j_0, k_0 \in I_n$; then the system has a unique solution for $n = 3$ and infinitely many solutions for $n > 3$ expressed in terms of $n - 3$ parameters.

2. If D_{ijk} does not exist or $D_{ijk} = 0$ for every $i, j, k \in I_n$; and there exists $D_{i_0 j_0 (n+1)} \neq 0$, $i_0, j_0 \in I_n$; then the system has no solution.

3. If D_{ijk} does not exist or $D_{ijk} = 0$ for every $i, j, k \in I_{n+1}$; then the solutions of the system coincide with the solutions of the system obtained by discarding of at least one of the equations. Then:

– 3.1. If the system contains contradictory equation or contradictory to each other equations, then the system has no solution.

– 3.2. If the system contains two equations in general position, then we discard the remaining equation and we solve the newly obtained system $2 \times n$ which has a unique solution for $n = 2$ and infinitely many solutions for $n > 2$, expressed through $n - 2$ parameters.

– 3.3. If one of the equations is regular, and the other equations are equivalent to it or zero, then the system reduces to the regular equation that has a unique solution for $n=1$ and infinitely many solutions for , expressed through $n-1$ parameter.

– 3.4. If the systems contains three zero equations, the set of solutions is $\square^{n}$, i.e. it is expressed through n parameters.

Proof. The case 1 contains the nonempty class 1 for systems in which $n \geq 3$. The case 2, the classes 2 – 4 (Misajleski et al. 2021). The case 3, contains the classes 5 – 21. The case 3.1, the classes 8, 9, 11 – 13, 15, 16, 18 – 20 (also applies to 2 and 3). The case 3.2, the classes 5 – 7. The classes 2 – 7 are non-empty for $n \geq 2$. Moreover, if the system contains a zero equation, it is rejected (5), if it contains two equivalent equations, one of them is rejected (6), otherwise, any equation is rejected (7). The case 3.3, the classes 10, 14 and 17; and the case 3.4, the class 21. So, all systems of equations are covered.

The order of the analyzes does not have to be as in Theorem 2.4.1. First we see if there exist zero equations (which are rejected) or contradictory equations (in which case the system has no solution). Then, among the regular equations, if there is a pair of contradictory to each other equations (then the system has no solution), or equivalent equations (then all equivalent equations except one are rejected). After these steps, which can be carried without writing, the system has no solution or is reduced to $2 \times n$ or is from geometric classes 1, 4 and 7. If the system is from the first Cramer class, then the solutions are obtained according to Cramer's formulas, if it is from the second (geometric class 4), it is concluded that the system has no solution, while if it is from the third (geometric class 7), any of the equations should be rejected.

Moreover, we will give examples of systems. We will not give examples for classes 1 and 4 because they are solved on standard way with determinants. For systems of class 2 and 3, there is no need to calculate determinants, because immediately by establishing the existence of contradictory or contradictory to each other equations, it is concluded that they have no solution.

Example 2.4.2. The determinants of the system $\begin{cases} x+y+z=2 \\ 3x-2y-4z=4 \\ 2x-3y-5z=2 \end{cases}$ are

$D = D_x = D_y = D_z = 0$. Moreover, the system does not contain zero, contradictory, a pair of mutually contradictory and equivalent equations (the system is of class 7), so we can discard any equation (for example the second one) and reduce it to a system 2×3 that will be solved by one of the standard methods, from where the solutions of the required 3×3 system are: $\left(\frac{8+2t}{5}, \frac{2-7t}{5}, t \right)$, for t real number.

In this system, if we saw the linear dependence of the equations, we did not have to calculate the determinants (this is not always obvious).

Example 2.4.3. The system of equations $\begin{cases} 4x-2y-2z=3 \\ 2x-y-z=1 \\ 6x-3y-3z=2 \end{cases}$ has no solution because the first and second equation are contradictory to each other ($\frac{4}{2}=\frac{-2}{-1}=\frac{-2}{-1}\neq\frac{3}{1}$). This system belongs to class 8.

Example 2.4.4. The three equations in the system $\begin{cases} x+2y+3z-u=8 \\ 2x+4y+6z-2u=16 \\ 3x+6y+9z-3u=24 \end{cases}$ are equivalent (the system belongs to class 10). It follows that the solutions of the system coincide with the solutions of the equation $x+2y+3z-u=8$ and given by $\left(8-2t-3k+l,t,k,l,\right)$, for t,k,l real numbers.

Example 2.4.5. The system of equations $\begin{cases} 0x+0y=1 \\ 2x-y=1 \\ 6x-3y=2 \end{cases}$ has no solution since the first equation is contradictory. The system belongs to class 11.

Example 2.4.6. The system $\begin{cases} 0x+0y+0z+0u+0v=8 \\ 2x+4y+6z-2u+2v=16 \\ 3x+6y+9z-3u+v=24 \end{cases}$ has no solution since the first equation is contradictory. The system belongs to class 2.

Example 2.4.7. The system of equations $\begin{cases} 2x+3y=1 \\ 2x-y=1 \\ 6x-3y=2 \end{cases}$ has no solution since the second and third equations are contradictory to each other ($\frac{6}{2}=\frac{-3}{-1}\neq\frac{2}{1}$). The system belongs to class 3.

Example 2.4.8. The first and third equations in the system $\begin{cases} 2x+0y+4z+2u=6 \\ 2x+0y+4z+u=7 \\ x+0y+2z+u=3 \end{cases}$ are equivalent. By discarding the first equation, the system reduces to the system of example 2.3.4. The system belongs to class 6.

The systems of the other classes either have no solution or are easily reduced to one equation whose solution is also easily determined.

3. Methodological significance of the results

In the secondary and especially in the university education in the Republic of N. Macedonia, the systems of linear equations in form of 2×2, 2×3, and 3×3, are studied via determinants (Celakoski et al. 2010; Misajleski 2018; Misajleski 2019; Trenchevski et al. 2018). Usually systems in which all determinants are zero are considered.

I. By naming the three classes of linear equations and the three mutual positions of a system of two regular linear equations, the systems are briefly discussed, i.e. the number of solutions is determined (in the case when the solutions are infinitely many, we know with how many parameters are expressed). Then the systems are classified and finally solved. Moreover, a systematic discussion of a system of two equations through the relations of their coefficients is given, which applies to all systems, not only to systems containing an equation with nonzero coefficients. Determinants have been successfully incorporated into three equation systems and this approach can be applied in the educational system when the determinants are studied.

II. The geometric interpretation of the equations in the system offers visualization of the mutual positions of the equations. Sketches of the classes for systems 2×2, 2×3, and 3×3 can be a useful tool for students to determine the various mutual positions of the equations.

III. Geometry class sketches can enable teachers to quickly form systems of equations that exhaust the various mutual positions of the equations.

IV. Greater visibility allows to perceive inaccuracies in the theory of systems of linear equations in school literature in the Republic of N. Macedonia. So for example for a system 2×2 in university textbooks (Georgievska et al. 2002) on page 18 – 19; (Shapkarev et al. 1995) on page 67 – 68; (Mitevska et al. 2006) on page 23, (Trenchevski et al. 2018) on page 15, as well as in other textbooks in the Republic of N. Macedonia, it is claimed that: "if $D = D_x = D_y = 0$ then the system has infinite solutions". But from the geometric interpretation, it can be seen that the claim is not valid in the case when one equation in the system is contradictory and the other is zero, or both equations are contradictory, which is not emphasized. In some literature such as (Malcheski 2007) on page 10, the statement is correctly formulated, since the contradictory equations are rejected and an example is given that the statement does not hold when one of the equations is contradictory. It is similar in (Misajleski 2018) – (Misajleski 2019). In (Celakoski et al. 2010) on page 26 are discusses the solutions of systems through the ratio of their coefficients, but it is not emphasized that the discussion does not apply to systems in which both equations have at least one zero coefficient.

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