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INTERESTING PROOFS OF SOME ALGEBRAIC INEQUALITIES

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Abstract. The paper considers interesting proofs of two algebraic inequalities.
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First, we will give the proof of the next algebraic inequality with radicals:

$$\sqrt{x^2+3}+\sqrt{y^2+3}+\sqrt{xy+3}\ge 6;\ \left(x,y\ge 0,\ x+y=2\right). \quad (1)$$

Proof: We will prove the two inequalities:

$$\sqrt{x^2+3}+\sqrt{y^2+3}\ge\sqrt{19-3xy} \quad (2)$$

and

$$\sqrt{19-3xy}+\sqrt{xy+3}\ge 6\ , \quad (3)$$

where $x,y\ge 0$ and $x+y=2$.

1^0. We have $0\le xy=t\le 1$ because $\frac{x+y}{2}\ge\sqrt{xy}$, i.e. $1\ge\sqrt{xy}\Rightarrow 0\le xy\le 1$. On the other hand

$$x^2+y^2=\left(x+y\right)^2-2xy=4-2t\ .$$

After squaring, we get from (2):

$$x^2+3+y^2+3+2\sqrt{x^2+3}\cdot\sqrt{y^2+3}\ge 19-3xy$$

$$\Leftrightarrow\left(x+y\right)^2-2xy+6+2\sqrt{x^2y^2+3\left(x^2+y^2\right)+9}\ge 19-3xy$$

$$\Leftrightarrow 4-2t+6+2\sqrt{t^2+3\left(4-2t\right)+9}\ge 19-3t$$

$$\Leftrightarrow 2\sqrt{t^2-6t+21}\ge 9-t$$

$$\Leftrightarrow 4\left(t^2-6t+21\right)\ge 81-18t+t^2$$

$$\Leftrightarrow 3t^2-6t+3\ge 0$$

$$\Leftrightarrow t^2-2t+1\ge 0$$

$$\Leftrightarrow (t-1)^2\ge 0,$$

which is obvious. Equality holds true for $t=1$, i.e. $xy=1\Rightarrow y=\frac{1}{x}\Rightarrow x+\frac{1}{x}=2\Rightarrow (x-1)^2=0\Rightarrow x=1$ and $y=1$. Thus, the inequality (2) is proved.

2^0. Now, we will prove the inequality (3), i.e.:

$$\sqrt{19-3xy}+\sqrt{xy+3}\ge 6$$

$$\Leftrightarrow 19-3xy+xy+3+2\sqrt{19-3xy}\cdot\sqrt{xy+3}\ge 36$$

$$\Leftrightarrow 2\sqrt{10xy+57-3x^2y^2}\ge 14+2xy$$

$$\Leftrightarrow \sqrt{10t+57-3t^2}\ge 7+t$$

$$\Leftrightarrow 10t+57-3t^2\ge 49+14t+t^2$$

$$\Leftrightarrow 4t^2+4t-8\le 0$$

$$\Leftrightarrow t^2+t-2\le 0$$

$$\Leftrightarrow (t-1)(t+2)\le 0,$$

which is true because $0\le t\le 1$, i.e. $t-1\le 0$. Equality holds, for $t=1$, i.e. for $x=y=1$. Thus, the inequality (3) is also proved.

The given inequality (1) follows from (2) and (3).

In the proof above a very important question arises: How to come to the inequality (2)? We will give an answer to this question in the proof of the next inequality:

$$\sqrt{x^2+8}+\sqrt{y^2+8}+\sqrt{xy+8}\ge 9;\ \left(x,y\ge 0,\ x+y=2\right). \qquad (4)$$

Proof: We will consider the inequality:

$$\sqrt{x^2+8}+\sqrt{y^2+8}\ge\sqrt{\alpha-\beta xy};\ \left(\alpha,\beta\in\mathbb{Q}\right) \qquad (5)$$

$$\Leftrightarrow x^2+8+y^2+8+2\sqrt{x^2+8}\cdot\sqrt{y^2+8}\ge\alpha-\beta xy$$

$$\Leftrightarrow x^2+y^2+16+2\sqrt{x^2y^2+8\left(x^2+y^2\right)+64}\ge\alpha-\beta xy$$

$$\Leftrightarrow 4-2t+16+2\sqrt{t^2+8(4-2t)+64}\ge\alpha-\beta t$$

$$\Leftrightarrow 2\sqrt{t^2-16t+96}\ge\alpha-20+(2-\beta)t\,. \tag{6}$$

Equality holds in (5) for $x=y=1$, and we get from (5) that $\sqrt{\alpha-\beta}=6$, i.e. $\alpha-\beta=36$.

The inequality (6) is equivalent now with the inequality:

$$2\sqrt{t^2-16t+96}\ge 16+\beta+(2-\beta)t$$

$$\Leftrightarrow 4\left(t^2-16t+96\right)\ge(16+\beta)^2+2(16+\beta)(2-\beta)t+(2-\beta)^2t^2\,.$$

Let now

$$P(t)=\left[4-(2-\beta)^2\right]t^2+\left[-64-2(16+\beta)(2-\beta)\right]t+384-(16+\beta)^2\ge 0\,.$$

Since $P(1)$ is the minimal value of the polynomial in the interval under consideration, then it follows that $P'(1)=0$, i.e.:

$$P'(t)=2\left[4-(2-\beta)^2\right]t-64-2(16+\beta)(2-\beta)$$

$$\Rightarrow P'(1)=8-2(2-\beta)^2-64-2(16+\beta)(2-\beta)=0$$

$$\Rightarrow 4-\left(4-4\beta+\beta^2\right)-32-\left(32-14\beta-\beta^2\right)=0$$

$$\Rightarrow 4-4+4\beta-\beta^2-32-32+14\beta+\beta^2=0$$

$$\Rightarrow 18\beta-64=0$$

$$\Rightarrow \beta=\frac{32}{9}\,,$$

and from here (because $\alpha-\beta=36$) we get $\alpha=\frac{356}{9}$.

We will now prove the inequality:

$$\sqrt{x^2+8}+\sqrt{y^2+8}\ge\sqrt{\frac{356}{9}-\frac{32}{9}xy} \tag{7}$$

$$\Leftrightarrow x^2+8+y^2+8+2\sqrt{x^2+8}\cdot\sqrt{y^2+8}\geq\frac{356}{9}-\frac{32}{9}xy$$

$$\Leftrightarrow 4-2t+16+2\sqrt{t^2-16t+96}\geq\frac{356}{9}-\frac{32}{9}t$$

$$\Leftrightarrow 9\sqrt{t^2-16t+96}\geq 88-7t$$

$$\Leftrightarrow 81\left(t^2-16t+96\right)\geq 7744-1232t+49t^2$$

$$\Leftrightarrow 32t^2-64t+32\geq 0$$

$$\Leftrightarrow t^2-2t+1\geq 0$$

$$\Leftrightarrow (t-1)^2\geq 0\ ,$$

which is obvious. Equality holds for $t=1$, i.e. $x=y=1$. Thus, the inequality (7) is proved.

Finally, we will prove the inequality:

$$\sqrt{\frac{356}{9}-\frac{32}{9}xy}+\sqrt{xy+8}\geq 9 \tag{8}$$

$$\Leftrightarrow \frac{356}{9}-\frac{32}{9}xy+xy+8+2\sqrt{\frac{356}{9}-\frac{32}{9}xy}\cdot\sqrt{xy+8}\geq 81$$

$$\Leftrightarrow 356-32xy+9xy+72+6\sqrt{(356-32xy)(xy+8)}\geq 729$$

$$\Leftrightarrow 428-23t+6\sqrt{100t-32t^2+2848}\geq 729$$

$$\Leftrightarrow 6\sqrt{100t-32t^2+2848}\geq 301+23t$$

$$\Leftrightarrow 36\left(100t-32t^2+2848\right)\geq 90601+529t^2+13846t$$

$$\Leftrightarrow 1681t^2+10246t-11927\leq 0$$

$$\Leftrightarrow (t-1)(1681t+11927)\leq 0\ ,$$

which is true because $0 \le t \le 1$, i.e. $t - 1 \le 0$. Equality holds for $t = 1$, i.e. $x = y = 1$. Thus, the inequality (8) is proved.

Now, the given inequality (4) follows from (7) and (8).

We leave the following inequality to the reader:

$$\sqrt{x^2 + 15} + \sqrt{y^2 + 15} + \sqrt{xy + 15} \ge 12; \ (x, y \ge 0, \ x + y = 2). \qquad (9)$$

For the proof of this inequality one could consider two other inequalities:

$$\sqrt{x^2 + 15} + \sqrt{y^2 + 15} \ge \sqrt{\alpha - \beta xy}; \ (\alpha, \beta \in \mathbb{Q}) \qquad (10)$$

and

$$\sqrt{\alpha - \beta xy} + \sqrt{xy + 15} \ge 12. \qquad (11)$$

Finally, we formulate a generalised version:

$$\sqrt{x^2 + n^2 - 1} + \sqrt{y^2 + n^2 - 1} + \sqrt{xy + n^2 - 1} \ge 3n \qquad (12)$$

where $x, y \ge 0, \ x + y = 2$ and $n \in \mathbb{N}, \ (n \ge 2)$.

Much work but at the same much pleasure. This is Mathematics.

NOTES/БЕЛЕЖКИ

1. AOPS-Art of ProblemSolving (web page).

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