Educational Pesearch
Научно–методически статии

HONEYCOMB – A GENIUS CREATION OF NATURE

Risto Malčeski
FON University, Skopje

Abstract. The paper considers how bees construct the cells of the beehive. It turns out that the constructions are subjected to strict mathematical laws, which makes them surprising masterpieces of nature.

Keywords: bee, cell, beehive, regular polygon, area.

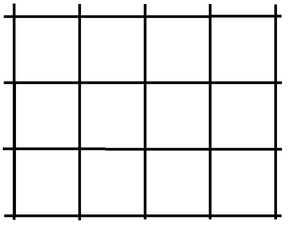


Figure 1

A variety of bees in a beehive, as a fully regulated society, is managed by certain laws. Each bee executes her job, which is strictly differentiated in the hive. None stands still. Not even those that seem to be suspended and peaceful, those which hang in the grapes. Even they do important work in the beehive. They produce wax that other bees use to build the honeycomb, which contains thousands of compartments and chambers and which in fact is a true masterpiece of nature. The following considerations will give us the reason of Maeterlinck's words about the hexagon cell of the honeycomb:

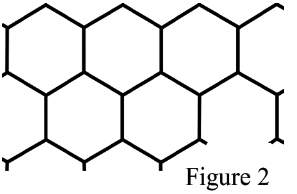


Figure 2

All geniuses together can do nothing to repair it. Neither any living being, nor a human have not done such a thing in their field of action, as the bees did.

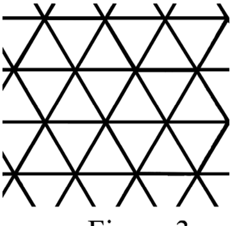


Figure 3

What has impressed Maeterlinck so much? The bees built four types of cells. In this paper we will discuss only the regular cells, i.e. the drone and the worker bee cells, because their sizes are consistent and their construction is calculated and precise. It seems that there is no better. Neverethe-less, we will see.

Even Pythagoreans knew the following fact: If you want to cover the plane with regular congruent polygons, the only possibilities are by the use of squares (Figure 1), regular hexagons (Figure 2) and equilateral triangles (Figure 3). But this obviously is known by the bees, so they cover the plane with regular congruent hexagons, i.e. the base of each cell is a regular hexagon. Clearly, it is important to cover the whole plane with no gaps, so that the neighbor cells could use the same cell walls. What enables the bees when building honeycomb to save precious wax and why regular hexagons, not squares or equilateral triangles are used? The answer to this question lies in the fact that out of the three types of regular polygons with the same perimeter, that can completely cover the plane with one type, the hexagons cover the largest area. This means that, when building a cell with a given volume, if the base is a regular hexagon, then less wax is used for the construction of the walls. Indeed, consider an equilateral triangle, a square and a regular hexagon with the same perimeters (Figure 4). Labeling their sides *a*, *b* and

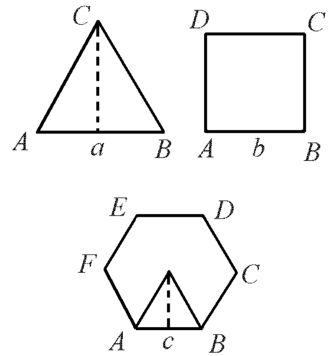


Figure 4

c, respectively, we get $3a = 4b = 6c$. So, $b = \frac{3}{4}a, c = \frac{1}{2}a$, and the areas of the polygons are given by the following formulae (Grozdev, 2007):

$$S_{\Delta ABC} = \frac{a^2\sqrt{3}}{4},\ S_{\square ABCD} = b^2 = \frac{9}{16}a^2,\ S_{ABCDEF} = \frac{3c^2\sqrt{3}}{2} = \frac{3a^2\sqrt{3}}{4}.$$

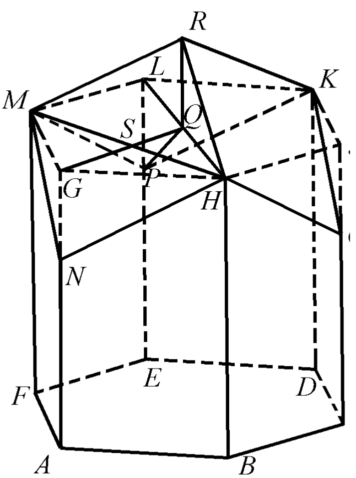


Figure 5

According to the inequality $\sqrt{3} < \frac{9}{4} < 3\sqrt{3}$ the following relation may be deduced $S_{\Delta ABC} < S_{\square ABCD} < S_{ABCDEF}$ for the areas of the polygons.

As we have already seen the cells completely cover the plane by regular hexagons and such coverage aims at saving wax. But the honeycomb is made in such a way that the cells are linked by the base to each other, so that the given capacity takes minimum area. Moreover, for the purpose of saving wax and strengthening, the cells are modified. Each cell of the honeycomb is a part of the hexagon prismatic space with the bottom restricted by three rhombi, and forms a three-faced pyramid. Hence, the surface areas form trapezoids (figure 5). Therefore, of particular importance are the sizes of the angles of the rhombi and the trapezoids. Depending on that, wax is saved during the construction of the cells. We also have to mention the way the cells are distributed into the space, i.e. the way the cells form the honeycomb. The honeycomb is made of cells, organized in two layers and the bases of one layer are placed under the other in a strictly determined way. The pyramid bottom of one cell in the front layer, consisting of three rhombi, acts as a part of the bases of the three cells of the opposite layer, wherein each of the three rhombi belongs to another cell of the opposite layer, thus forming ⅓ of its "base". In such a manner the bees not only save wax, placing the cells in the honeycomb with no gaps but also the mentioned placement (distribution) receives advantages regarding the strength of the construction. Namely, the common edge of two lateral faces of a cell ends in a vertex in which rhombi of this cell meet, so the honeycomb gains strength.

Our further focus of interest will be the angles between the planes of the cells as well as the angles in the rhombi. The main goal of doing this is to use wax as least as possible, i.e. the area of the cells to be minimal.

Let us consider figure 5, MH and GQ are diagonals of the rhombus $MGHQ$, so $\Delta MGH \cong \Delta MQH$ and $\overline{GS} = \overline{SQ}$. Similarly, MH and RN are diagonals of the rhombus $NMRH$, so $\overline{SR} = \overline{SN}$ and $\Delta MNH \cong \Delta MRH$. Further, $\measuredangle RSQ = \measuredangle NSG$, as cross-angles. Therefore, $\overline{GS} = \overline{SQ}$, $\overline{SR} = \overline{SN}$ and $\measuredangle RSQ = \measuredangle NSG$, so $\Delta NSG \cong \Delta RSQ$, i.e. $\overline{RQ} = \overline{NG}$. We have already shown $\Delta MGH \cong \Delta MQH$ and $\overline{RQ} = \overline{NG}$. Now, let us consider the prisms $NHMG$ and $MHRQ$. According to the last, it follows that $V_{NHMG} = V_{MHRQ}$. Analogous considerations apply to the other two honeycomb rhombi. So we get that for any angle which occupy the rhombi $NHRM$, $OKRH$ and $PMRK$ the capacity of the honeycomb is equal to the capacity of the hexagonal prism whose second base is the hexagon $GHJKLM$.

From the above it follows that we have to determine only the position of the rhombus in such a way that the area of the honeycomb cell is minimal (the capacity of the honeycomb cell is constant). Letting, $\overline{AB} = \overline{GH} = a$, $\overline{AG} = h$ and $\overline{GN} = x$, we get $S_{ABHN} = \frac{h+h-x}{2} a = ah - \frac{ax}{2}$, and further, the area of a cell face is

$$S_1 = 6S_{ABHN} = 6ah - 3ax . \tag{1}$$

Now, we have to determine the area of the rhombus $MNHR$. It is sufficient to determine the length of its diagonals MH and RN. Using the fact that, ΔGHQ is an equilateral triangle with side a and SH is its altitude, we get $\overline{SQ} = \frac{a}{2}$ and $\overline{SH} = \frac{a\sqrt{3}}{2}$. Using the Pithagoras theorem for the triangle ΔSGN we get the following

$$\overline{SN} = \sqrt{(\tfrac{a}{2})^2 + x^2} = \sqrt{x^2 + \tfrac{a^2}{4}} .$$

So, $\overline{MH} = 2\overline{SH} = a\sqrt{3}$ and we compute the area of rhombus $MNHR$:

$$S_{MNHR} = \frac{\overline{MH} \cdot \overline{RN}}{2} = \overline{MH} \cdot \overline{SN} = a\sqrt{3} \cdot \sqrt{x^2 + \tfrac{a^2}{4}} = \sqrt{3a^2x^2 + \tfrac{3a^4}{4}} .$$

Therefore, the area of the pyramidal part of the honeycomb cell is given by

$$S_2 = 3S_{MNHR} = 3\sqrt{3a^2x^2 + \frac{3a^4}{4}}\,. \tag{2}$$

Finally, (1) and (2) imply that the area of the honeycomb cell is as follows

$$S = S_1 + S_2 = 6ah - 3ax + 3\sqrt{3a^2x^2 + \frac{3a^4}{4}}\,. \tag{3}$$

So, we should determine the minimum of the function (3), i.e. we have to determine x such that the function S gets its minimum. From the obvious inequality

$$(x - \frac{a}{2\sqrt{2}})^2 + \frac{a^2}{8} \geq 0 \tag{4}$$

we get the following equivalent inequalities (the following inequalities are equivalent to the given one)

$$x^2 - \frac{2ax}{2\sqrt{2}} + \frac{a^2}{8} + \frac{a^2}{8} \geq 0,$$

$$2x^2 + \frac{a^2}{2} \geq \frac{2ax}{\sqrt{2}},$$

$$2x^2a^2 + \frac{a^4}{2} \geq \frac{2a^3x}{\sqrt{2}},$$

$$3x^2a^2 + \frac{3a^4}{4} \geq x^2a^2 + \frac{2a^3x}{\sqrt{2}} + \frac{a^4}{2},$$

$$3x^2a^2 + \frac{3a^4}{4} \geq (ax + \frac{a^2}{\sqrt{2}})^2,$$

$$\sqrt{3x^2a^2 + \frac{3a^4}{4}} \geq ax + \frac{a^2}{\sqrt{2}},$$

$$6ah - 3ax + 3\sqrt{3x^2a^2 + \frac{3a^4}{4}} \geq 6ah + \frac{3a^2}{\sqrt{2}},$$

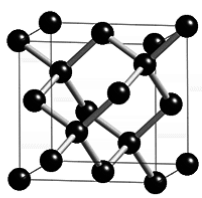
It means that for each x the values of the function S are greater or equal to $6ah + \frac{3a^2}{\sqrt{2}}$ and $S_{\min} = 6ah + \frac{3a^2}{\sqrt{2}}$ if and only if the right hand side of (4) reaches its minimum, i.e. if and only if $x = \frac{a}{2\sqrt{2}}$.

Further, using the value of x we get $\overline{SN} = \sqrt{(\frac{a}{2\sqrt{2}})^2 + \frac{a^2}{4}} = \frac{a\sqrt{3}}{2\sqrt{2}}$ and moreover, for $\overline{SH} = \frac{a\sqrt{3}}{2}$ we get $\operatorname{tg}\measuredangle SNH = \frac{\overline{SH}}{\overline{SN}} = \sqrt{2}$, i.e. $\measuredangle SNH = 54^0\ 44'8''$. So, we can compute the angles $\measuredangle MNH = 2\measuredangle SNH = 109^0\ 28'16''$ and $\measuredangle NMR = 180^0 - 109^0\ 28'16'' = 70^0\ 31'44''$. These are more precise values of the angles obtained by Maraldy and Kasni since 1712 when they directly measured them. In 1712 Maraldy and Kasni measured those angles to be $109^0\ 28'$ and $70^0\ 32'$, respectively. Let us consider the angle between the base of the pyramid and the rhombus *MRHN*, i.e. the angle $\measuredangle GNS$. We get, $\overline{GN} = x = \frac{a}{2\sqrt{2}}$ and $\overline{GS} = \overline{SQ} = \frac{a}{2}$, so $\operatorname{tg}\measuredangle GNS = \frac{\overline{GS}}{\overline{GN}} = \sqrt{2}$ and that is the angle which Maraldy and Kasni obtained when they measured it directly.

So, we may conclude that bees make their honeycomb using minimum amount of wax as least as possible. Of course, we cannot say that bees use complex mathematical operations while building their honeycomb, however, we may simply say that the honeycomb is a miracle of nature.

A population is very similar to the bee one, namely axles also make hexagonal honeycomb cells, but in this case there is no double layer of cells. So, the same bottom is not shared among several cells, which gives special strength to the honeycomb. Nevertheless, does only the shared bottom of the honeycomb provide hardness strength or something else is the real reason for it? Let us consider the vertices M, H, K and R. It follows from the above, that:

$\overline{KM} = \overline{KH} = \overline{MH} = 2\overline{SH} = a\sqrt{3}$ and



$$\overline{RK} = \overline{RM} = \overline{RH} = \sqrt{\overline{SH}^2 + \overline{SR}^2} = \sqrt{\frac{3a^2}{4} + \frac{3a^2}{8}} = \frac{3a}{2\sqrt{2}}.$$

Letting $a\sqrt{3} = b$, we get

$$\overline{KM} = \overline{KH} = \overline{MH} = b \text{ и } \overline{RK} = \overline{RM} = \overline{RH} = \frac{b\sqrt{3}}{2\sqrt{2}}.$$

It means that the vertices M, H and K belong to a regular tetrahedron with edge length equal to b, and R is center of the sphere drawn around this tetrahedron. Is this a coincidence? The answer is not known yet, but it is enough to observe that the carbon atoms in diamond, which has the highest known hardness in nature (see the picture) are deployed in exactly the same way.

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✉ **Prof. Dr. Risto Malčeski**
Faculty of Informatics
FON University
Skopje, Republic of Macedonia
E-mail: risto.malceski@gmail.com