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GEOGEBRA IMPACT IN AVOIDING COMMON MISTAKES STUDENTS MAKE IN HANDLING EXPONENTIAL FUNCTIONS

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Abstract. Exponential functions, symbolized by $f(x) = a^x$, represent fundamental mathematical concepts with extensive applications in real-world scenarios, ranging from population growth to compound interest and radioactive decay (Tuda & Rexhepi 2023). This abstract highlights the significance of GeoGebra, an intuitive mathematical software, in interactive visualization and analysis, particularly in addressing common mistakes students make when engaging with exponential functions. Through its user-friendly interface, GeoGebra facilitates a hands-on approach to learning, offering students a practical and engaging environment to rectify misconceptions and deepen their understanding of the mathematical principles governing exponential functions.

This study incorporates a comparative analysis of traditional and technological teaching methods, focused on 2nd-year students at high school "8-Shtatori" in Skopje. By evaluating the efficacy of both approaches in fostering a deeper comprehension of exponential functions, the outcomes contribute valuable insights into pedagogical strategies, shedding light on optimizing the learning experience for students grappling with exponential functions. Additionally, surveys and direct comparisons of solutions with students in the classroom setting provided firsthand observations on how learners interact with the material and identified common mistakes made during problem-solving. The insights gained from these surveys and discussions further informed the development and refinement of the approach, ensuring a comprehensive understanding of both the benefits and challenges associated with learning exponential functions through GeoGebra.

Keywords: exponential functions; analysis; students; GeoGebra

1. Introduction

Exponential functions, embodied in the expression $f(x) = a^x, a > 0, a \neq 1$, serve as a fundamental and versatile concept in mathematics, finding applications in various real-world phenomena such as population growth, compound interest, and radioactive decay. As we delve into this paper, our focus extends beyond the inherent properties of exponential functions to address the common mistakes students often encounter when navigating these dynamic mathematical models.

In this paper, we draw attention to the seminal work of (Redlin 2018), which underscores the practical utility of exponential models in predicting population sizes, calculating investment returns, and determining the remaining amounts of radioactive substances. Expanding upon this foundational understanding, our paper incorporates insights gained from a visit to school "8-Shtatori" in Skopje, providing valuable firsthand experiences into the challenges students face when attempting to grasp exponential functions, thereby guiding our emphasis on rectifying common misconceptions.

This paper emphasizes the vital role of GeoGebra, a dynamic mathematical software, in aiding comprehension and visualization of exponential functions, while also addressing and rectifying the errors frequently made by students. We draw inspiration from (Kamberi et al. 2022), who recognize the transformative impact of internet resources and engaging visual materials in enhancing understanding of mathematical concepts. Additionally, we acknowledge the findings of (Mollakuqe et al. 2021), who note that while students in lower classes may initially find GeoGebra challenging, its visual and interactive nature significantly aids comprehension, particularly as students progress to higher classes.

The exponential function stands as a mathematical bedrock with far-reaching implications across various domains. Originating from the concept of exponentiation, its modern definition extends its applicability to real, complex numbers, and beyond, embracing diverse mathematical objects. While the theoretical understanding of exponential functions is essential, their practical significance lies in their ability to model dynamic processes and phenomena observed in the real world. In this study, we delve into the exploration of exponential functions, aiming to not only uncover their inherent properties but also to showcase their practical applications and relevance in everyday life. Through the integration of technology, particularly GeoGebra software, we aim to provide a comprehensive understanding of exponential functions and their real-world implications.

The integration of GeoGebra serves as a cornerstone in our approach, leveraging its intuitive platform to address the challenges highlighted during the visit to "8-Shtatori". We acknowledge the findings of (Mollakuqe et al. 2021), who noted that while students in lower classes may initially find GeoGebra challenging, its visual and interactive nature significantly aids comprehension, particularly as student's progress to higher classes.

Through GeoGebra, we seek to empower students, educators, and researchers alike to not only explore and visualize exponential functions but also rectify common mistakes, fostering a deeper and more accurate understanding of these vital mathematical concepts.

2. Research method

1. In the initial phase of this research, an exhaustive review of existing literature on exponential functions will be undertaken. The primary focus will be on delineating foundational concepts, exploring real-world applications, and identifying prevalent mistakes made by students. Works by scholars, including (Redlin 2018), (Kamberi et al. 2022), and (Mollakuqe et al. 2021), will be scrutinized to establish a comprehensive and theoretical foundation for this study. This literature review aims to provide a solid framework for understanding the current state of knowledge in the field, informing subsequent stages of the research methodology.

2. Observational visit to school “8-Shtatori” in Skopje: As part of this research, an observational visit to “8-Shtatori” in Skopje will be conducted to gain firsthand insights into the challenges students face when dealing with exponential functions. Engagements with educators and students during this visit will provide qualitative data on the specific mistakes and misconceptions prevalent in the educational context of “8-Shtatori”. This observational component aims to enrich the research by incorporating real-world perspectives into the study of common mistakes associated with exponential functions.

3. Interviews and Surveys: To complement the observational findings, interviews will be conducted with educators at school “8-Shtatori” to gather qualitative insights into their teaching methods and perceptions of student challenges with exponential functions. Additionally, surveys will be administered to students, aiming to collect quantitative data on their experiences, understanding, and potential mistakes related to exponential functions. The combination of interviews and surveys will contribute to a comprehensive understanding of both qualitative and quantitative aspects of student challenges.

4. Integration of GeoGebra: GeoGebra, a dynamic mathematical software, will be integrated as an instructional tool during this research paper. The impact of GeoGebra on student learning will be evaluated through a combination of surveys, direct observations, and assessments. This aspect of the research focuses on assessing how GeoGebra aids in rectifying common mistakes made by students and enhancing their overall understanding of exponential functions.

5. Analysis of Student Work: Samples of student work related to exponential functions will be collected and analyzed. The analysis will involve categorizing and examining the types of mistakes commonly made by students, with a focus on identifying recurring patterns and trends. Student

work, including homework assignments, quizzes, and tests, will serve as valuable sources of data for understanding the specific challenges students face in comprehending exponential functions.

6. Comparative Study: A comparative analysis will be conducted to assess the effectiveness of both traditional teaching methods and the integration of GeoGebra in addressing common mistakes associated with exponential functions. Quantitative data from surveys and assessments will be utilized to measure improvements in student comprehension, providing insights into the comparative efficacy of each approach.

7. Data Analysis: Quantitative and qualitative data collected throughout the research will undergo rigorous statistical analysis. The analysis aims to identify correlations, patterns, and trends related to student mistakes, GeoGebra usage, and overall learning outcomes. This systematic approach will enhance the robustness of the research findings.

3. Exponential function

The exponential function, denoted as:

$$f(x) = \exp(x) \text{ or } e^x \tag{1}$$

where x is represented as an exponent, is a fundamental mathematical concept. This positive-valued function of a real variable extends its applicability to complex numbers and various mathematical objects such as matrices or Lie algebras. Initially rooted in the concept of exponentiation, the modern definition allows for rigorous extension to all real arguments, including irrational numbers.

The significance of the exponential function in pure and applied mathematics is profound, leading mathematician Walter Rudin to assert that it is "the most important function in mathematics" (Rudin 1987). This function finds broad applications owing to its versatility and prevalence in mathematical models.

The exponential function is expressed in the form:

$$f(x) = a.b^x \tag{2}$$

where:

• $f(x)$ denotes the function's value at a specific point x on the real number line;

• a is a constant known as the "initial value" or "y-intercept", representing the function's value when $x = 0$;

• b is a constant referred to as the "base" of the exponential function. It is a positive real number greater than 0 and not equal to 1;

• x serves as the independent variable, capable of assuming any real number.

4. Comparative Study: Traditional vs. Technological Teaching Methods in school "8-Shtatori" – Skopje

Together with 2nd-year high school students, we delved into the fascinating world of exponential functions, utilizing a range of teaching methods to enhance their understanding and engagement. The central focus of our session was to compare the effectiveness of the traditional classical method of teaching and the use of GeoGebra software.

Below, we presented mathematical tasks related to exponential functions using both approaches, allowing these 2nd-year high school students to experience both the conventional and technological ways of learning. This collaborative endeavor not only enriched the students' understanding of exponential functions but also highlighted the practical implications of incorporating technology into mathematical exploration.

5. Analyzing exponential functions: a comparative study of classical and GeoGebra solutions

Task 1: Modeling Population Growth (fig. 1).

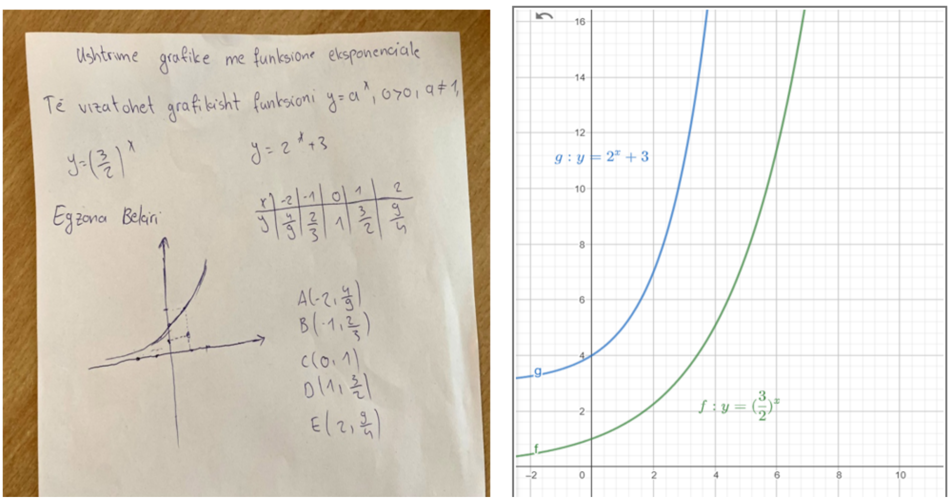

Figure 1. Task 1 — classical and GeoGebra solution

Objective: Explore how exponential functions can be used to model population growth and understand its implications.

Implementation:

Pen-and-Paper Exploration:

Understanding the Model: Students analyzed the exponential function $y = \left(\frac{3}{2}\right)^x$ as a model for population growth.

Algebraic Analysis: Through manual calculations, students determined the behavior of the function and its implications for population growth.

Graphical Representation: Students graphed the function manually, visualizing how the population grows exponentially over time.

GeoGebra Analysis:

Dynamic Visualization: Transitioning to GeoGebra software, dynamically visualized the population growth model, observing how changes in parameters affect population dynamics.

Interactive Exploration: With GeoGebra, students interactively explored the graph of the exponential function, observing how slight adjustments in parameters affect its shape and behavior.

Comparison of Approaches: By comparing the manual and GeoGebra approaches, students gained a comprehensive understanding of the exponential function $y = \left(\frac{3}{2}\right)^x$, merging classical problem-solving techniques with modern technological tools.

Interactive Exploration: Students interactively explored the graph, adjusting parameters to simulate different scenarios and understand the impact on population growth.

Real-World Applications: Discussed real-world examples where exponential functions are used to model population growth, such as demographic studies and urban planning.

Results:

Pen-and-Paper Findings: Students gain insights into how exponential functions can model population growth and understand the implications for real-world scenarios.

GeoGebra Observations: Utilizing GeoGebra, students experience the dynamic visualization of population growth, facilitating a deeper understanding of exponential functions and their practical applications.

Task 2: Exploring the Exponential Function $y = 2^x$ (fig. 2).

Objective: Investigate the behavior and properties of the exponential function $y = 2^x$ through both traditional pen-and-paper methods and GeoGebra software.

Implementation:

Pen-and-Paper Exploration:

Analysis of Key Properties: Students analyzed the exponential function $y = 2^x$ using pen and paper, focusing on understanding its key properties such as domain, range, and behavior.

Algebraic Manipulation: Through manual solution methods, students refined their algebraic skills by solving equations involving the exponential function $y = 2^x$.

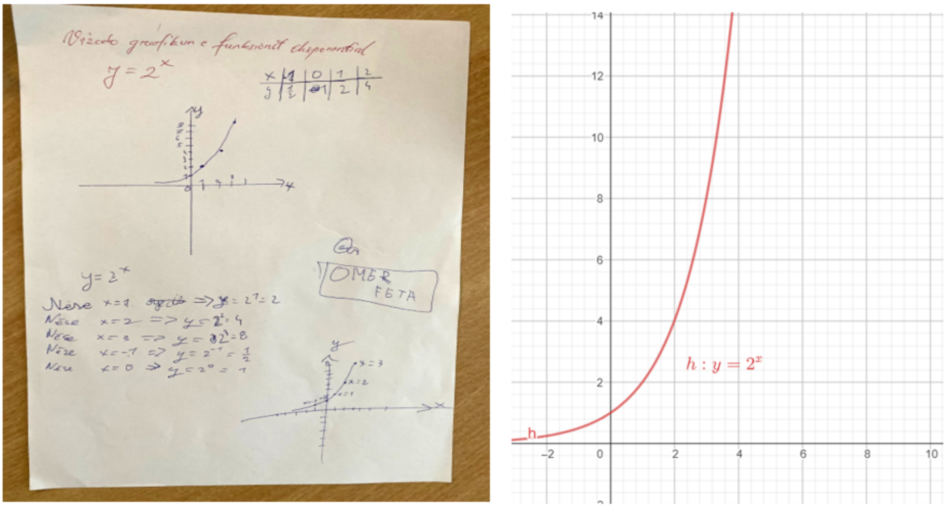

Figure 2. Task 2 – classical and GeoGebra solution

Graphical Representation: Students graphed the function $y = 2^x$ manually, gaining insights into its graphical behavior, including exponential growth.

GeoGebra Analysis:

Visualization with GeoGebra: Transitioning to GeoGebra software, students utilized its dynamic visualization capabilities to graph the function $y = 2^x$ digitally.

Interactive Exploration: With GeoGebra, students interactively explored the graph of the exponential function, observing how changes in parameters affect its shape and behavior.

Comparison of Approaches: By comparing the manual and GeoGebra approaches, students gained a comprehensive understanding of the exponential function $y = 2^x$, merging traditional problem-solving techniques with modern technological tools.

Results:

Pen-and-Paper Findings: Through manual exploration, students gained insights into the algebraic properties and graphical behavior of the exponential function $y = 2^x$. They developed proficiency in solving equations involving exponential functions and understanding their fundamental properties.

GeoGebra Observations: Utilizing GeoGebra, students experienced the dynamic visualization of the function's graph, enabling interactive exploration and deeper comprehension. They observed how slight adjustments in parameters lead to significant changes in the graph, reinforcing their understanding of exponential growth.

Task 3: Exploring Exponential Functions with Different Bases (fig. 3).

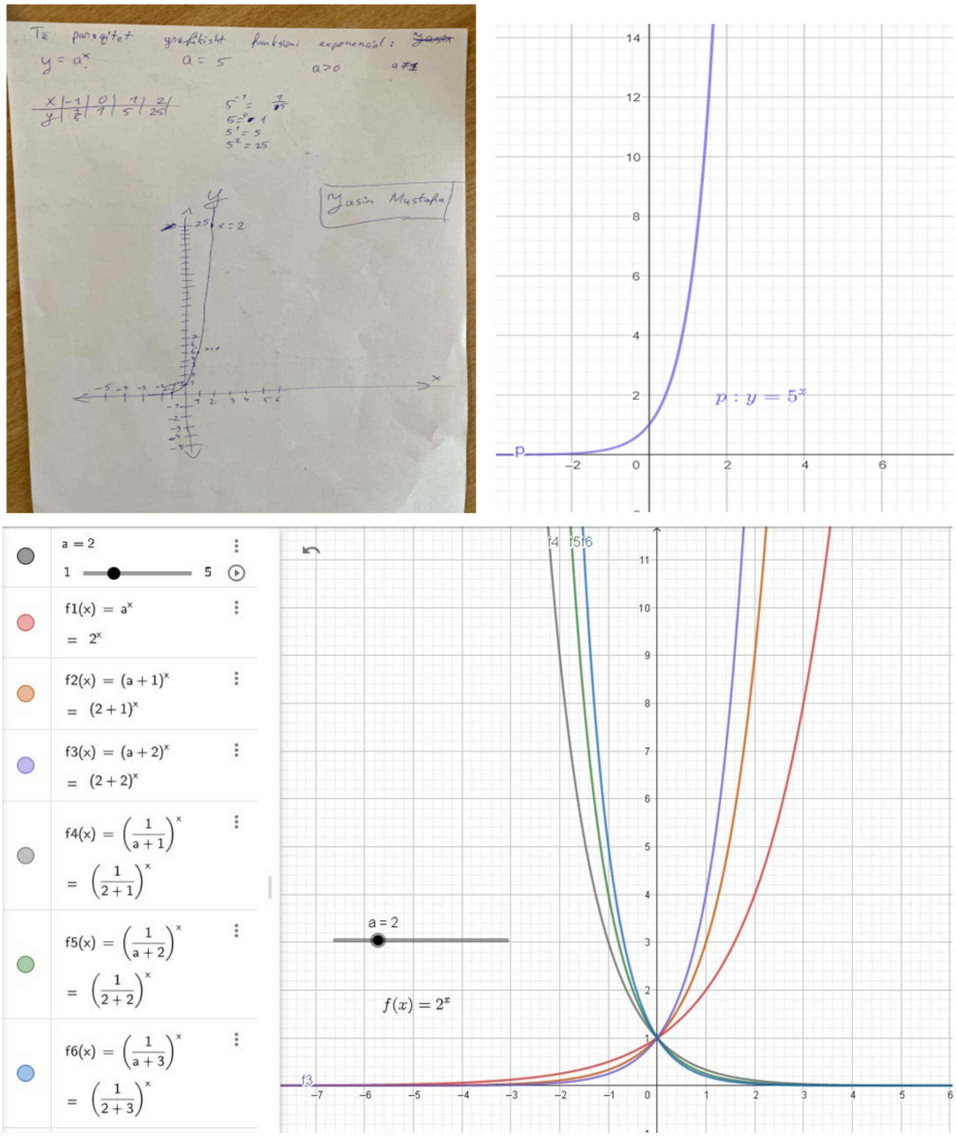

Figure 3. Task 3 – classical and GeoGebra solution

Function: $y = a^x$.
Implementation:
Pen-and-Paper Exploration:
Identify Key Properties: Students analyzed the general exponential function $y = a^x$, recognizing that the base a determines the nature of exponential growth or decay.
Variation of Bases: Students considered different values for a to observe how the base influences the shape and behavior of the exponential function. This allowed them to understand the concept of exponential growth or decay across various bases.
GeoGebra Analysis:
Transition to GeoGebra: After exploring the general exponential function on pen and paper, students utilized GeoGebra software to draw the graphs of different exponential functions $y = a^x$, with different bases.
Visualization and Comparison: With GeoGebra, students graphically compared multiple functions such as $y = 2^x, y = 3^x, y = 5^x$, and their inverses. This visual comparison highlighted how different bases influence the function's shape and rate of change.
Results:
Pen-and-Paper Findings: Students understood how altering the base a affects the exponential function's growth or decay. They observed diverse patterns based on the base's value, noting variations in the curve's steepness and direction.
GeoGebra Observations: In GeoGebra, we explored various exponential functions with a dynamic slider to vary the base. By drawing the graphs of functions such as $y = 2^x, y = 3^x, y = 5^x, y = (1/2)^x, y = (1/3)^x$,and $y = (1/5)^x$, students observed the following:
• Increasing Functions: The graphs exhibited rapid growth with increasingly steep curves as the base increased. Functions with larger bases grew more quickly, illustrating exponential growth vividly.
• Decreasing Functions: Conversely, functions $y = (1/2)^x, y = (1/3)^x$, and $y = (1/5)^x$ showed an exponential decrease. The curves flattened as x increased, with the decay becoming slower for functions with smaller bases.
This practical example allowed students to visually compare the effects of different bases on exponential growth and decay, enhancing their understanding of how the base influences the function's behavior.

6. Common mistakes in handling exponential functions

After solving the task involving the exponential function $y = (2/3)^x$, students were asked to identify common mistakes in handling exponential functions. The task provided an opportunity for students to apply their understanding of exponential functions.

The common mistakes identified, as seen in the figure above and other mistakes of the students, as they were discussed as teachers, we dictated to the students to write as comments where the mistakes are made most often, as written by the students in the accompanying photo/figure (fig.4), include:

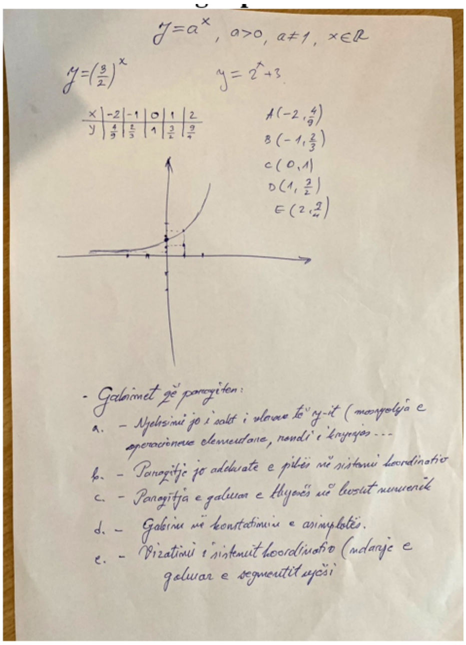

Figure 4

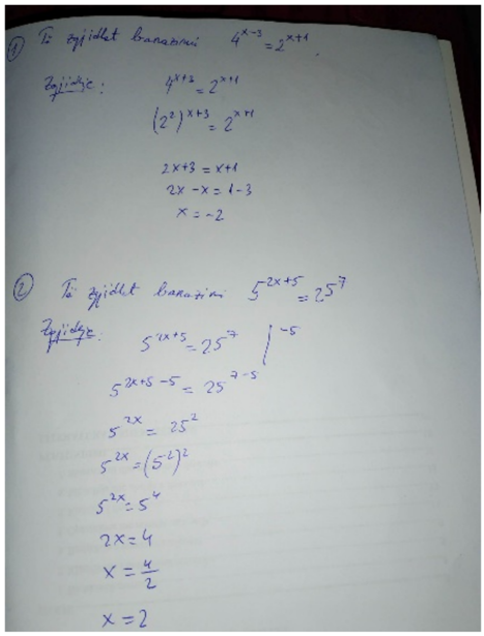

Figure 5

• inaccurate computation of y-values (lack of understanding of basic mathematical operations, order of operations);
• inadequate representation of a point in the coordinate system;
• incorrect representation of the axis intercept in the numerical axis;
• errors in identifying asymptotes;
• incorrect drawing of the coordinate system (incorrect division of the segment).

These observations offer valuable insights into areas where students may need additional support and guidance in mastering exponential functions. By addressing these common mistakes, students can enhance their understanding and proficiency in handling exponential equations effectively.

In the following, we present some of the most frequent mistakes that students make when solving exponential equations.

During the experience in the classroom, we noticed that these mistakes are mostly made by forgetting the properties of exponents, which according to the curriculum in North Macedonia are taught in the first year, while the exponential function is in the third year. For this reason, for the first hours of the exponential function, we suggest repeating the properties of exponents and solving different tasks with them.

In the first task in fig. 5 student multiplies the exponent 2 by x only, which is a mistake. The exponent 2 must be multiplied by the binomial $x+2$, while in the second task operations on exponents are done without making sides with the same bases.

As mentioned above, it is forgetting the rules of exponents. The students usually compare the obtained results with the software and see that they do not correspond. GeoGebra in this case only prompts the students to ask questions since the mistake was made unconsciously.

7. Overcoming common mistakes in exponential functions

After pinpointing common mistakes made by students in handling exponential functions, our next crucial step was to engage in a constructive discussion on how to circumvent these errors. In the discussion, we included:

• Enhancing y-Value Recognition: Emphasized the importance of mastering basic operations and the correct order of operations. Provided targeted practice exercises to reinforce these fundamental mathematical concepts.

• Improving Point Representation: Introduced visualization techniques and practice exercises to refine skills in presenting points graphically.

• Correcting Axis Intercept Representation: Clarified the proper methods for representing intercepts on the numerical axis.

Provided examples and walkthroughs to ensure students grasped the correct procedures.

• Refining Asymptote Identification: Offered additional exercises specifically focused on identifying asymptotes. Encouraged the use of multiple approaches to confirm the accuracy of asymptote determinations.

• Ensuring Accurate Coordinate System Drawing: Guided students in the correct division of segments when drawing the coordinate system. Facilitated interactive sessions where students can apply these principles in real-time.

8. Results and discussion

In order to gather insightful feedback, we administered a survey to our students, probing their experiences and perceptions regarding the use of GeoGebra handling with exponential functions. The survey comprised two key questions aimed at evaluating the effectiveness of GeoGebra in aiding the visualization and computation of exponential functions.

Survey questions:

1. Did GeoGebra make it easier for you to visualize and study exponential functions?

2. Did using GeoGebra assist you in accurately computing y-values, particularly in understanding basic mathematical operations?

Table 1. Answers of the two questions

Options	1^{st} question	2^{nd} question
Yes	19	16
No	4	7

Among the responses received for the first question, 19 students answered affirmatively, indicating that GeoGebra indeed eased the process of visualizing and plotting exponential functions. On the contrary, 4 students reported that they did not find GeoGebra helpful in this regard.

In response to the second question, 16 students acknowledged that GeoGebra aided them in accurately computing y-values, especially concerning basic mathematical operations. However, 7 students expressed that they did not observe a significant improvement in this aspect.

The survey findings revealed a notable consensus among students regarding the benefits of GeoGebra, particularly in the context of understanding exponential functions. Many students lauded the dynamic interface of GeoGebra, which allowed them to visualize and interact with exponential concepts in ways surpassing traditional methods. The software's ability to dynamically illustrate various parameters, generate custom graphs, and offer real-time feedback received widespread praise from the students.

Moreover, the survey responses underscored that GeoGebra not only made complex mathematical concepts more accessible but also fostered a sense of engagement and curiosity among students. The hands-on experience with GeoGebra was credited with significantly enhancing students' understanding of exponential growth compared to conventional classroom approaches.

In conclusion, the unanimous agreement among students on the efficacy of GeoGebra for understanding exponential functions highlights the software's invaluable contribution to the learning experience. It serves as an enriching and empowering tool for mathematical exploration, enabling students to grasp complex concepts with greater ease and engagement.

9. Tasks Description

Duration: The tasks were conducted over a one-hour session, focusing on exploring exponential functions using both traditional pen-and-paper methods and GeoGebra software.

Scope: The tasks aimed to compare the effectiveness of traditional pen-and-paper methods with GeoGebra software in teaching exponential functions to second-year high school students. Objectives included introducing the concept of exponential functions, providing hands-on exploration, identifying common mistakes, and evaluating student understanding.

Preparation: Prior to the session, materials including printed worksheets and GeoGebra software were prepared. The classroom was arranged for collaborative learning, and instructions for using GeoGebra were provided.

Data Collection: Data were collected through observation, task completion, surveys, and discussions during the session. Feedback from students was gathered to assess understanding, preferences, and experiences.

Reporting: Following the session, a report summarized findings, observations, and student feedback, highlighting the effectiveness of GeoGebra in enhancing student engagement and understanding of exponential functions.

10. Conclusion

In conclusion, the exponential function stands as a mathematical bedrock with far-reaching implications across various domains. Originating from the concept of exponentiation, its modern definition extends its applicability to real, complex numbers, and beyond, embracing diverse mathematical objects.

As we explore exponential functions, it becomes evident that their influence extends beyond mere mathematical abstraction. They offer a powerful tool for modeling dynamic processes, providing insights into real-world phenomena. The integration of tools like GeoGebra further enhances the educational aspect, offering an interactive platform for students to visualize and understand exponential functions.

This journey into the realm of exponential functions has unveiled not only their theoretical underpinnings but also the practical applications and the challenges students often face. By delving into the common mistakes encountered by students and leveraging educational tools, such as GeoGebra, we aim to enhance the learning experience and foster a deeper understanding of this fundamental mathematical concept.

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