FIVE WAYS TO SOLVE A PROBLEM FOR A TRIANGLE

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Abstract. A triangle with the property $3\alpha + 2\beta = 180^0$ is considered in the paper. Five different ways are proposed to prove that $a^2 + bc = c^2$. The paper is of methodological character.

Keywords: triangle, problem solving, trigonometry, similarity.

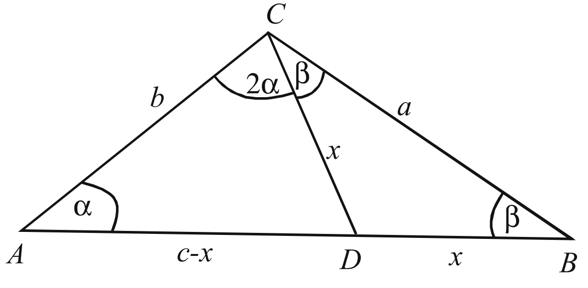
To solve a mathematical problem in several ways is challenging and creative but also instructive and useful for a deep examination of the problem history and content to discover the initial idea of its author and to find the potential of the possible applications. This is exceptionally impotant for talented students who touch various mathematical domains by the different solutions, thus increasing their knowledge and capacity. Consider the following problem:

Using the standart notations for ΔABC*, prove that* $a^2 + bc = c^2$ *if* $3\alpha + 2\beta = 180^0$.

We will give five solutions of this problem.

Solution 1. Accounting for the relations $3\alpha + 2\beta = 180^0$ and $\alpha + \beta + \gamma = 180^0$, we have $\gamma = 180^0 - (\alpha + \beta) = 3\alpha + 2\beta - (\alpha + \beta) = 2\alpha + \beta$. It follows that $\gamma > \beta$ and hence $c > b$. Take now a point D on the side AB such that $\angle BCD = \angle ABC = \beta$ (Fig. 1). This implies that $BD = AD$, i.e.

Fig. 1



ΔBCD is isosceles. If the lenghts of the legs are denoted by x, then $AD = c - x$. By the law of sines and the law of cosines for ΔADC we get:

$$\frac{x}{\sin\alpha} = \frac{c-x}{\sin 2\alpha} \text{ and } x^2 = b^2 + (c-x)^2 - 2b(c-x)\cos\alpha .$$

Since $\sin 2\alpha = 2\sin\alpha\cos\alpha$, the first equality implies that $\cos\alpha = \frac{c-x}{2x}$. Now, the second equality becomes $x^2 = b^2 + (c-x)^2 - 2b(c-x)\frac{c-x}{2x}$ and consequently $x = \frac{c^2}{b+2c}$. The law of sines for ΔBCD and the law of cosines for ΔABC give

$$\frac{x}{\sin\beta} = \frac{a}{\sin(180^0 - 2\beta)} \text{ and } b^2 = a^2 + c^2 - 2ac\cos\beta, \text{ respectively.}$$

Since $\sin(180^0 - 2\beta) = \sin 2\beta = 2\sin\beta\cos\beta$, it follows from the first equality that $\cos\beta = \frac{a}{2x}$. Now, the second equality becomes $b^2 = a^2 + c^2 - 2ac\frac{a}{2x}$ and consequently $x = \frac{a^2c}{a^2+c^2-b^2}$. We have obtained 2 expresions for x, which give:

$$\frac{c^2}{b+2c} = \frac{a^2c}{a^2+c^2-b^2} .$$

It is easy to check that the last is equivalent to $a^2 + bc = c^2$.

Solution 2. Take a point E on the side AB such that $\angle ACE = \angle CAE = \alpha$ (Fig.2).

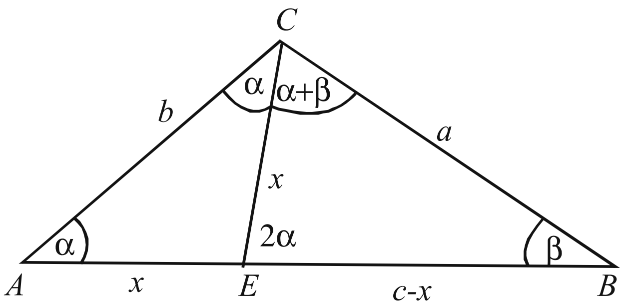


Fig. 2

Now ΔAEC is isosceles and if $AE = CE = x$, then $BE = c - x$. Applying the Mollweid' s formula for ΔBCE, we get:

$$\frac{(c-x)+x}{a}=\frac{\cos\frac{(\alpha+\beta)-\beta}{2}}{\sin\frac{2\alpha}{2}}=\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{1}{2\sin\frac{\alpha}{2}}.$$

Thus $\frac{c}{a}=\frac{1}{2\sin\frac{\alpha}{2}}$ and consequently $\sin\frac{\alpha}{2}=\frac{a}{2c}$. Further, apply the the law of cosines for ΔABC:

$$a^2=b^2+c^2-2bc\cos\alpha=b^2+c^2-2bc\left(1-2\sin^2\frac{\alpha}{2}\right)=$$
$$=b^2+c^2-2bc\left(1-\frac{a^2}{2c^2}\right)=b^2+c^2-\frac{b}{c}(2c^2-a^2).$$

Now it is easy to check that the equality $a^2=b^2+c^2-\frac{b}{c}(2c^2-a^2)$ is equivalent to $a^2+bc=c^2$, using that $c>b$.

Solution 3. Take a point F on the line AC (C is between A and F) such that $\angle CBF=\alpha+\beta$. (Fig. 3)

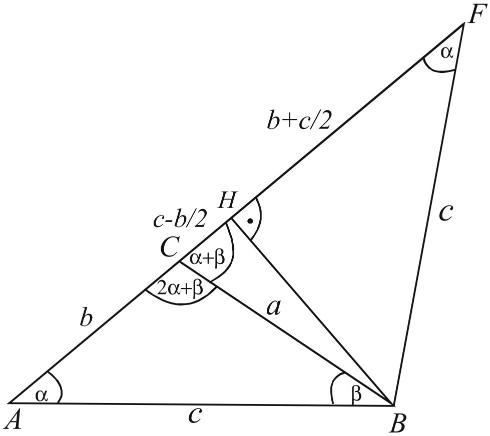


Fig. 3

Since $\angle BCF=\alpha+\beta$, then ΔBCF is isoceles. Also, we have that $\angle BFC=\alpha$ and it follows that ΔABF is isoceles too, i.e. $AB=BF=CF=c$. If BH is the height of ΔABF ($H\in AF$), then $AH=\frac{1}{2}AF=\frac{b+c}{2}$ and consequently

$CH = \frac{b+c}{2} - b = \frac{c-b}{2}$, using again that $c > b$. Finally, apply the Pithagoras theorem to the right triangles ΔBHF and ΔBCH. We have:

$$BH^2 = c^2 - \left(\frac{b+c}{2}\right)^2 \text{ and } BH^2 = a^2 - \left(\frac{c-b}{2}\right)^2.$$

Now, check that the equality $c^2 - \left(\frac{b+c}{2}\right)^2 = a^2 - \left(\frac{c-b}{2}\right)^2$ is equivalent to $a^2 + bc = c^2$.

Solution 4. Applying the Stewart' s theorem to ΔABF (Fig. 3), we have:

$AF \cdot (AC \cdot CF + BC^2) = AB^2 \cdot CF + BF^2 \cdot AC$, which gives, that

$(b+c)(bc+a^2) = c^2c + c^2b$, i.e. $(b+c)(bc+a^2) = (c+b)c^2$.

The last ie equivalent to $a^2 + bc = c^2$.

Solution 5. Let $G \in AB$ be such point that $\angle BCG = \angle BAC = \alpha$. (Fig. 4)

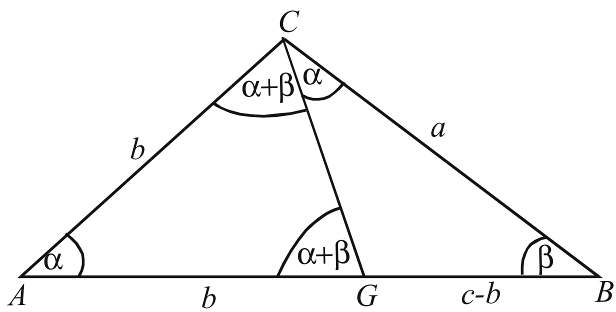


Fig. 4

It follows that $\angle AGC = \angle ACG = \alpha + \beta$ and ΔAGC is isosceles. Hence, $GB = c - b$. Since $\angle GCB = \alpha$, the triangles ABC and CBG are similar. Thus, $\frac{a}{c-b} = \frac{c}{a}$, which is equivalent to $a^2 + bc = c^2$.

Using trigonometry (the law of sines, the law of cosines and the Mollweid's formula), solution 1 and solution 2 are rather complicated. On the contrary, solutions 3–5 seem to be shorter, avoiding trigonometry. They apply the Pythagoras theorem, the Stewart' s theorem and similarity of triangles, which are deep mathematical facts. Maybe, this makes them more interesting and creative.

For exercise we recommend the following problems:

1. Given is ΔABC. Prove that the relations $\alpha = 2\beta$ and $a^2 = b(b+c)$ are equaivalent.

2. If the relation $\alpha - \beta = 90^0$ is valid for ΔABC, prove that $(a^2 - b^2)^2 = c^2(a^2 + b^2)$. (Such a triangle is called *pseudoright triangle*).

3. The angles α, β and γ of ΔABC satisfy the equality $2\gamma = \alpha - \beta$. Prove that $c^2 = a(a-b)$.

4. The proportion $\alpha : \beta : \gamma = 4:2:1$ is valid for ΔABC. Prove that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.

5. It is given for an acute ΔABC that $\angle ACB = 2\angle ABC$. If $D \in BC$ is such a point that $2\angle BAD = \angle ABC$, prove that $\frac{1}{BD} = \frac{1}{AB} + \frac{1}{AC}$.

6. If the equality $\alpha = 3\beta$ is valid for ΔABC, prove that $bc^2 = (a-b)^2(a+b)$.

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ПЕТ НАЧИНА ЗА РЕШАВАНЕ НА ЕДНА ЗАДАЧА ЗА ТРИЪГЪЛНИК

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Резюме. В статията се разглежда триъгълник със свойството $3\alpha + 2\beta = 180^0$. Предложени са пет начина за доказване, че $a^2 + bc = c^2$. Статията е с методически характер.

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