EMPIRICAL BAYESIAN ESTIMATES OF OPERATIONAL RELIABILITY RELATED TO ELECTRONIC ITEMS FOR MEDICAL PURPOSE

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Abstract. This paper is focused on a reliability analysis of the exponential model parameters by a reliability parametric Bayesian estimation in regard to one type of medical items. The analysis is based on using a priori data for analyzing current incoming information. The ŧ case study aims at formalizing prior and accumulated sampling data obtained from operational events occurring during the operation in the form of reliability empirical data with respect to a type of semi-automatic blood pressure monitors. As a result of the case study and the analysis, the point estimates of the operational reliability indices valid for the medical items under study are experimentally obtained.

Keywords: E-Bayesian point estimates, parametric Bayesian approach, reliability data analysis, type I censored data

Introduction

In providing high reliability of electronic items a wide range of issues arise and dealing with them is most often based on experience, observation and experiment. This creates opportunity for developing a specific mathematical model on which to base the actions for optimization involved in the maintenance of these items (Georgiev et al., 2016; Garipova et al., 2018; Garipova, 2017). The acquisition, procession and analysis of data regarding these items performance under real operational conditions play an important role for this purpose.

The mathematization of human knowledge on a global scale has been developed for centuries, and the intensity of research in this area has increased sharply over the last decades (Georgiev, 2016; Georgieva & Georgiev, 1999; Georgiev et al., 2013; Nikolov et al., 2016). One possible tool for reliability analysis is the empirical Bayesian assessment. Bayesian theorems form the methodological basis in the transition process from a priori information (formalized in the form of a priori distribution) to the posterior information. This process is a sequential accumulation of data information. At the initial stage, the collected experience (analogous research information from the past) regarding the properties and performance of the tested electronic items is studied. New information in the form of empirical data, which commonly differs from the priori information, is obtained during the test. Thereby, a gradually reconsidering and reassessment of the prior data is feasible. At any time, it is possible to give a generalized description of the item properties and parameters and this description is comprehensive and complete as it is based on all the existing information up to date. This process is continuous and it continues after receiving the next data set for ongoing experiment.

The subject under study in this paper is the operational reliability of a set of electronic items through a parametric empirical Bayesian assessment regarding statistical data obtained from real service containing items maintenance and repair. The operational reliability assessment analysis is based on pre-accumulated priori information and after a certain period of time - on obtaining new data set information in the form of empirical data. This enables the development and optimization of algorithms of stochastic mathematical models which consider the maintenance data.

Maximum likelihood estimation

Let t_1, t_2, ..., t_n are the sampling data from life testing regarding operational events which occurred during the operation of nn electronic items. In the reliability theory, the distribution of the random variable time for the first failure occurrence in the flow is following the exponential law. Based on the no-effect property in the elementary flow, the mean time between failures (*MTBF)* of repairable electronic items is also exponential. Therefore, the failure time of an electronic item is exponentially distributed with probability density function (pdf)

$$f\left(t_i\right) = \lambda e^{-\lambda t_i} \tag{1}$$

The total likelihood is the product of the independent pdf's:

$$L(\lambda) \equiv f(t_i \mid \lambda) = \lambda^n e^{-\lambda \sum_i t_i} \quad (2)$$

Let assume that a set of n items is under observation for up to time t_0 and n_f of these items have demonstrated a failure with corresponding failure times $t_1, t_2, ..., t_{nf}$. The other $n_w = n - n_f$ items are still functioning. The incomplete data (i.e. during the test, not all the items have failed and so their failure times are unknown) is called censored. In this case, the data are right (type I) censored. For each item

$$\begin{aligned} f(T > t_0) &= f(\text{still working at time } t_0) \\ &= \int_{t_0}^{\infty} \lambda e^{-\lambda t} \\ &= \left[-e^{-\lambda t}\right]_{t_0}^{\infty} = e^{-\lambda t_0} \end{aligned} \quad (3)$$

(this is the Poisson probability for no events with mean λt_0. In other words, this is the contribution to the likelihood at time t_0 from functioning items. With regard to failed items, the failure time t_1 is known and hence, the total likelihood, if $t_i = t_0$ is defined for the functioning items, can be calculated as follows[1)]:

$$\begin{aligned} L(\lambda) &= \lambda^{n_f} e^{-\lambda \sum_{i=1}^{n_f} t_i} \left(e^{-\lambda t_0}\right)^{n_w} \\ &= \lambda^{n_f} e^{-\lambda \sum_{i=1}^{n} t_i} \end{aligned} \quad (4)$$

Therefore, the maximum likelihood estimator (*MLE*) λ for λ is calculates as shown below:

$$\begin{aligned} &\frac{\partial L}{\partial \lambda} = n_f \lambda^{n_f - 1} e^{-\lambda \sum_{i=1}^{n} t_i} - \lambda^{nf} \left(\sum_{i=1}^{n} t_i\right) e^{-\lambda \sum_{i=1}^{n} t_i} = 0 \\ &\Rightarrow \quad n_f \hat{\lambda}^{n_f - 1} = \hat{\lambda}^{n_f} \left(\sum_{i=1}^{n} t_i\right) \quad \Rightarrow \quad \hat{\lambda} = \frac{n_f}{\sum_i t_i} \end{aligned} \quad (5)$$

Empirical Bayesian estimation (e-Bayesian estimation)

In estimation and decision theory, E-Bayesian estimation is based on applying the conditional probabilities in the modeling of the process for integration of available prior data accumulated from current tests with actual data information collected on a specific current test (Georgiev & Georgiev, 2017).

The diagram in Fig.1 illustrates the general Bayesian procedure in graphical form. The item properties are expressed in terms of the failure rate λ. The initial notion regarding this properties is based on some already known (prior) information E. The formalization of this prior data is performed by depicting the prior distribution of the parameterλ, which is conditional with respect to E, i.e. $\pi(\lambda | E)$.

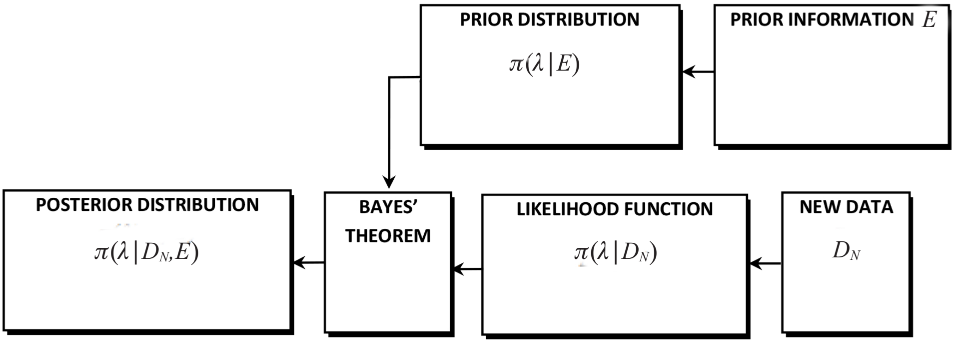

Figure 1. General E-Bayesian procedure

Let t_1, t_2, ..., t_n are the sampling data. For Bayesian estimator it is necessary to specify the prior distribution for the parameter λ. The conjugate prior on λ is modelled using a Gamma distribution having pdf

$$\pi(\lambda | E) = \frac{\beta^{\alpha} \lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda} \tag{6}$$

with hyperparameters E (α, β), where α is the number of total failures occurred in β time intervals.

The new empirical data D_N received in the testing process are recorded in a formalized form through the likelihood function $L(\lambda | D_N)$. The likelihood function is expressed as a probability density function for the realization of these empirical data and is recorded as a conditional function of the parameter λ. To obtain $L(\lambda | D_N)$ it is necessary to have knowledge concerning the probability model. This model most often is expressed as a conditional distribution of basic random variable (failure rate λ). Applying Bayes theorem for combining Eqs (6) and (4), the posterior of λ given t is calculated as follows

$$\pi(\lambda \mid D_N, E) = \frac{\pi(\lambda) L(E \mid \lambda)}{\int_0^\infty \pi(\lambda) L(E \mid \lambda) d\lambda}$$

$$= \frac{\frac{\beta^\alpha \lambda^{\alpha-1} \lambda^{n_f}}{\Gamma(\alpha)} e^{-\beta\lambda} e^{-\lambda \sum_{i=1}^{n} t_i}}{\int_0^\infty \frac{\beta^\alpha \lambda^{\alpha-1} \lambda^{n_f}}{\Gamma(\alpha)} e^{-\beta\lambda} e^{-\lambda \sum_{i=1}^{n} t_i} d\lambda} . \quad (7)$$

$$= \frac{\lambda^{\alpha-1+n_f} e^{-\lambda\left(\beta + \sum_{i=1}^{n} t_i\right)}}{\int_0^\infty \lambda^{\alpha-1+n_f} e^{-\lambda\left(\beta + \sum_{i=1}^{n} t_i\right)} d\lambda}$$

The denominator is obtained by means of the identity $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ and using the change of the variable $u = \lambda(\beta + \sum_{i=1}^{n} t_i)$. This results in $\lambda = \frac{u}{\beta + \sum_{i=1}^{n} t_i}$, and $d\lambda = \frac{du}{\beta + \sum_{i=1}^{n} t_i}$ with the limits of u same as λ. Substituting back into the posterior equation gives:

$$\pi(\lambda \mid D_N, E) = \frac{\lambda^{\alpha-1+n_f} e^{-\lambda\left(\beta + \sum_{i=1}^{n} t_i\right)}}{\frac{1}{\beta + \sum_{i=1}^{n} t_i} \int_0^\infty \left(\frac{u}{\beta + \sum_{i=1}^{n} t_i}\right)^{\alpha-1+n_f} e^{-u}\, du} . \quad (8)$$

$$= \frac{\lambda^{\alpha-1+n_f} e^{-\lambda\left(\beta + \sum_{i=1}^{n} t_i\right)}}{\frac{1}{\left(\beta + \sum_{i=1}^{n} t_i\right)^{\alpha+n_f}} \int_0^\infty u^{\alpha-1+n_f} e^{-u}\, du}$$

If $z = \alpha + n_f z = \alpha + n_f$, then

$$\pi(\lambda \mid E) = \frac{\lambda^{\alpha-1+n_f} e^{-\lambda\left(\beta+\sum_{i=1}^{n} t_i\right)}}{\frac{1}{\left(\beta+\sum_{i=1}^{n} t_i\right)^{\alpha+n_f}} \int_0^\infty u^{z-1} e^{-u}\, du}. \tag{9}$$

Using the identity $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$, the following expression is derived:

$$\pi(\lambda \mid E) = \frac{\lambda^{\alpha-1+n_f}\left(\beta+\sum_{i=1}^{n} t_i\right)^{\alpha+n_f}}{\Gamma\left(\alpha+n_f\right)} e^{-\lambda\left(\beta+\sum_{i=1}^{n} t_i\right)} \tag{10}$$

Let $\alpha' = \alpha + n_f \alpha' = \alpha + n_f$ and $\beta' = \beta + \sum_{i=1}^{n} t_i$:

$$\pi(\lambda \mid E) = \frac{\lambda^{\alpha'-1} \beta'^{\alpha'}}{\Gamma(\alpha')} e^{-\beta'\lambda} \tag{11}$$

As can be seen, the posterior is a Gamma distributed parameter with hyperparameters $\alpha' = \alpha + n_f$, and $\beta' = \beta + \sum_{i=1}^{n} t_i$. Therefore, the prior and posterior are of the same form, and Bayes' rule does not need to be re-calculated for each update (O'Connor et al., 2011; Deodatis et al., 2013). Instead the parameters can be simply update with new evidence. The E-Bayesian point estimates are presented below

$$\hat{\lambda}' = \frac{a'}{\beta'}, \quad Var\left(\hat{\lambda}'\right) = \frac{a'}{\beta'^2}. \tag{12}$$

Case study

Consider Table 1 representing statistical data regarding a number of 30 pieces of semi-automatic blood pressure monitors (*SABPMs*) of identical model. Among 30 pieces *SABPMs*, an amount of approximately 43% demonstrates a failure. For the purpose of empirical reliability analysis, it is assumed that all items are brought into a normal (useful) life at the same point in time.

Table 1. Statistical data regarding semi-automatic blood pressure monitors

№	Product	Model	Serial number	Adoption date	Transmission date	Guarantee	Status	Comment
1	Microlife	BPA50	241301***	26.01.2015	26.01.2015	13.01.2014	prevention	cleaning of PRV, testing – without deviation
2	Microlife	BPA50	421301***	03.06.2015	03.06.2015	17.09.2014	test	testing – without deviation
3	Microlife	BPA50	241301***	03.06.2015	03.06.2015	13.01.2014	prevention	cleaning of PRV, testing – without deviation
4	Microlife	BPA50	491302***	04.06.2015	04.06.2015	12.06.2014	prevention	operating instructions testing – without deviation
5	Microlife	BPA50	511200***	03.07.2015	03.07.2015	20.09.2013	repair	cleaning ribbon cable, testing -without deviation
6	Microlife	BPA50	301404***	31.08.2015	31.08.2015	12.05.2015	prevention	operating instructions, cleaning of PRV, testing – without deviation
7	Microlife	BPA50	511200***	31.08.2015	31.08.2015	20.09.2013	repair	replacement crystal oscillator testing – without deviation
8	Microlife	BPA50	391401***	30.10.2015	30.10.2015	20.10.2015	repair	adjusting the PRV, testing – without deviation
9	Microlife	BPA50	191406***	30.10.2015	30.10.2015	13.01.2015	test	testing – without deviation
10	Microlife	BPA50	351302***	08.12.2015	08.12.2015	06.03.2014	prevention	cleaning of PRV, testing – without deviation
11	Microlife	BPA50	241301***	23.12.2015	23.12.2015	21.11.2014	test	unfounded claims, testing – without deviation
12	Microlife	BPA50	301403***	07.01.2016	07.01.2016	15.05.2015	prevention	cleaning of PRV, testing – without deviation
13	Microlife	BPA50	391400***	07.01.2016	07.01.2016	22.12.2015	test	testing – without deviation
14	Microlife	BPA50	391400***	07.01.2016	07.01.2016	21.12.2015	test	testing – without deviation

15	Microlife	BPA50	481402***	12.01.2016	12.01.2016	24.12.2015	test	testing – without deviation
16	Microlife	BPA50	491302***	27.01.2016	27.01.2016	12.06.2014	prevention	cleaning of PRV, testing – without deviation
17	Microlife	BPA50	161302***	25.02.2016	25.02.2016	21.11.2013	repair	soldering the battery terminals testing – without deviation
18	Microlife	BPA50	491303***	14.03.2016	14.03.2016	15.01.2016	test	testing – without deviation
19	Microlife	BPA50	391400***	14.03.2016	14.03.2016	17.02.2016	test	testing – without deviation
20	Microlife	BPA50	161505***	25.04.2016	25.04.2016	18.01.2014	test	testing – without deviation
21	Microlife	BPA50	301402***	27.05.2016	27.05.2016	12.06.2015	test	testing – without deviation
22	Microlife	BPA50	341400***	20.06.2016	20.06.2016	13.12.2015	test	testing – without deviation
23	Microlife	BPA50	191406***	20.06.2016	20.06.2016	13.01.2015	test	testing – without deviation
24	Microlife	BPA50	101400***	20.06.2016	20.06.2016	29.06.2014	prevention	cleaning of ADV, testing – without deviation
25	Microlife	BPA50	191403***	20.06.2016	20.06.2016	03.12.2014	prevention	cleaning of ADV, testing – without deviation
26	Microlife	BPA50	341400***	20.06.2016	20.06.2016	27.08.2015	test	testing – without deviation
27	Microlife	BPA50	481403***	20.06.2016	20.06.2016	01.04.2016	test	testing – without deviation
28	Microlife	BPA50	421301***	25.08.2016	25.08.2016	01.04.2014	repair	replacement arm cuff, testing – without deviation
29	Microlife	BPA50	241301***	31.08.2016	31.08.2016	23.01.2014	test	testing – without deviation
30	Microlife	BPA50	391400***	15.12.2016	15.12.2016	22.12.2015	test	testing – without deviation

Let the statistical data from Table 1 is presented as sampling data regarding operational events that occurred during the operation of 30 *SABPMs* ($n = 30$). As can be seen, 13 of all measuring items are demonstrated a failure with corresponding failure times $t_1, t_2, \dots, t_{n_f}$ ($n_f = 13$). The order 17 items are still functioning ($n_w = 17$). Concerning the failed items, the failure times $t_1, t_2, \dots, t_{n_f}$ for each item are evaluated from the statistical data. Therefore, assuming that the observed time period is equal to the failure time ($t_0 = t_i$) and by means of (4), the likelihood function can be found (Fig.2).

The E-Bayesian point estimation regarding failure rate can be evaluated by Eq. (5) as follows:

$$\hat{\lambda} = \frac{n_f}{\sum_i t_i} = 1,53191.10^{-5} \text{ item/hour}.$$

Consider sampling data accumulated regarding *SABPMs* life testing in Table 1 and the subsequent data from Table 2 related to the same type SABPMs. The life of these items has an exponential distribution. For the purposes of E-Bayesian estimation, the data from both tables can be presented as a priori and posteriori data respectively.

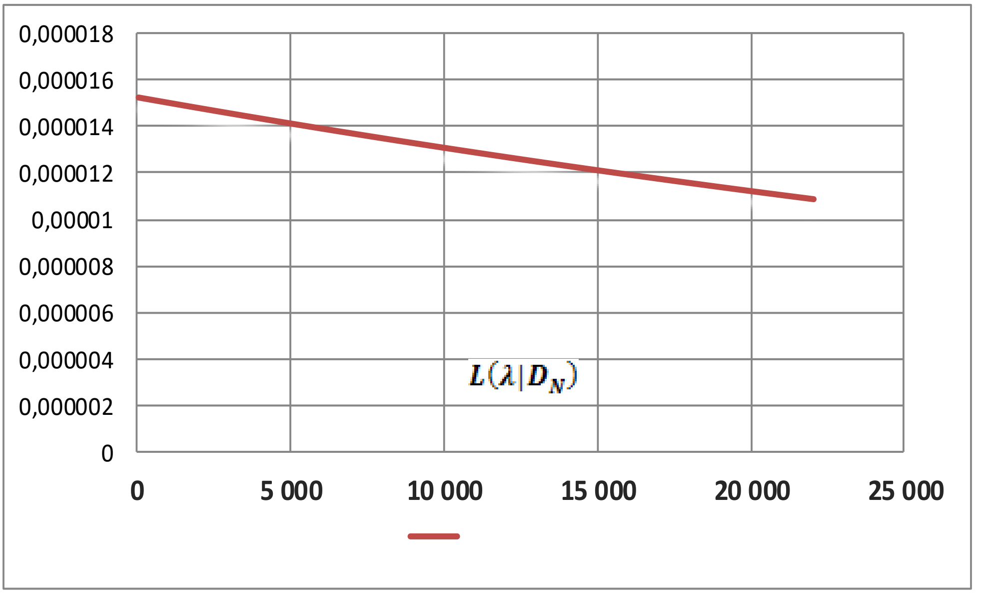

Figure 2. Likelihood function of the *SABPMs*

Table 2. Statistical data regarding semi-automatic blood pressure monitors

№	Product	Model	Serial number	Adoption date	Transmission date	Guarantee	Status	Comment
1	Microlife	BPA50	481400***	02.02.2017	02.02.2017	29.07.2016	test	testing – without deviation
2	Microlife	BPA50	391400***	09.03.2017	09.03.2017	02.12.2015	test	testing – without deviation
3	Microlife	BPA50	301400***	24.03.2017	24.03.2017	14.01.2015	test	testing – without deviation
4	Microlife	BPA50	391401***	28.03.2017	28.03.2017	12.10.2015	test	replacement of batteries, testing – without deviation
5	Microlife	BPA50	321601***	03.04.2017	03.04.2017	22.03.2017	test	testing – without deviation
6	Microlife	BPA50	191406***	12.04.2017	12.04.2017	28.11.2014	test	testing – without deviation
7	Microlife	BPA50	241301***	26.04.2017	26.04.2017	21.11.2014	prevention	cleaning of PRV, testing – without deviation
8	Microlife	BPA50	301403***	04.07.2017	04.07.2017	11.09.2015	test	testing – without deviation
9	Microlife	BPA50	421301***	04.08.2017	04.08.2017	11.05.2014	test	testing – without deviation

For this case, the conjugate prior on λ is modelled by means of Eq. (6). In relation to prior data from Table 1, the hyperparameter α is equaled to the total number of failures occured from all the previous data, and the hyperparameter β is equaled to the total of all the failure hours in previous test, i.e. $\alpha = 13$ and $\beta =$163896. The expression of Gamma posterior p.d.f. is given by Eq. (11) with updated hyperparameters $\alpha' = \alpha + n_f$ and $\beta' = \beta + \sum_{i=1}^{n} t_i$. From Table 2 the actual hyperparameter values are evaluated as $\alpha' = 14$ and $\beta' = 185184$. The both of obtained conjugate prior and posterior p.d.f. of the *SABPMs* under test are presented in Fig.3. The conjugate prior and posterior pdf's can be used for predictions. As can be seen, the prior pdf is much lower than pdf. In Fig. 4 the number of hours is set to 80000. The decreasing functions both of prior and posterior distributions show that approximately at the time the priori and posteriori probability are equal.

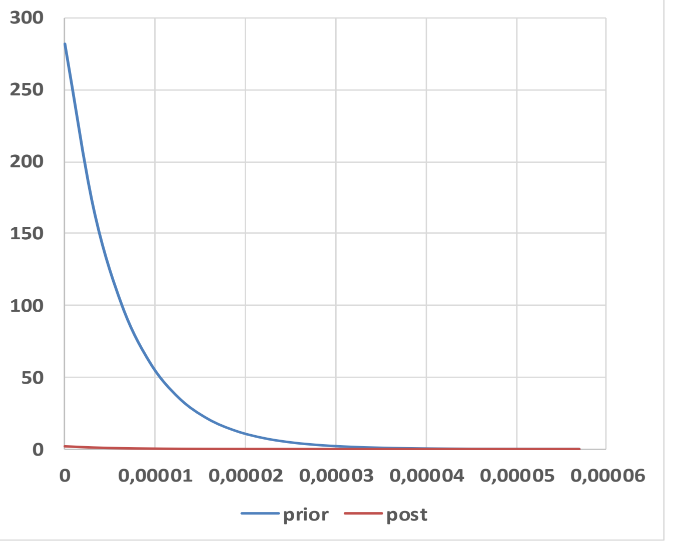

Figure 3. Prior and posterior distribution on the tested electronic items

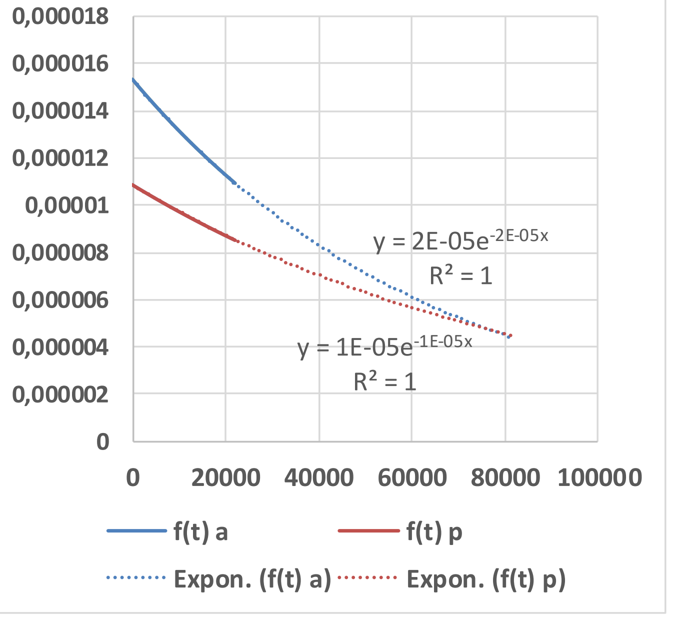

Figure 4. Predictive function of the prior and posterior pdf

The E-Bayesian point estimate values by means of Eq. (12) are presented below

$$\hat{\lambda}' = 7,56005.10^{-5} \quad Var\left(\hat{\lambda}'\right) = 4,08245.10^{-5}.$$

Final remarks

This paper discusses the application of Bayesian techniques for reliability prediction of electronic items during the life testing period. It is assumed that the time till failure of these items follows an exponential distribution. A posterior distribution for the failureless work probability is then derived using Jeffreys improper prior and methods for predicting the equipment reliability are suggested and demonstrated. The nature of present author's attitude to the prediction issue is that no confidence limits are required.

The presented concept regarding the operational reliability of the medical items observed is based on the reliability indices assessment through the MLE and parametric E-Bayesian approach as well. Life testing of electronic items associated with exponential distribution and Gamma distribution is discussed. The authenticity of the empirical values obtained is achieved using statistical modeling methods. The E-Bayesian estimation is a flexible approach that allows the constantly updating of a priori data through accumulating new data.

NOTES

1. https://cosmologist.info/teaching/STAT/Slides.pdf

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