Educational Technology
Образователни технологии

EDUCATION OF A MATHEMATICIAN – EXPERIMENTALIST, OR SOFT MANIFESTO OF EXPERIMENTAL MATHEMATICS

[1)]**Alexander Yastrebov,** [2)]**Maria Shabanova**
[1)]*Yaroslavl State Pedagogical University*
[2)]*Institute of Mathematics, Information and Space Technologies*

Abstract. This article is written under the influence of V. I. Arnold's works, in which he suggested the possibility of considering mathematics as an experimental science, and proved the existence of phenomena in the description of which "soft" models have an advantage over the "hard". The article is a kind of response to the manifesto, which was delivered by R. V. Shamin, head of the group of experimental mathematics. By this manifesto the group declares the emergence of a new branch in mathematics, which is characterized by the extensive use of computer experiments. Due to the fact that the means and methods of experimental mathematics have lately become increasingly involved in teaching mathematics, the authors of this article have considered it necessary to have their own manifesto published. The article aims to remind readers what place was occupied and is occupied by experimental methods in mathematics and mathematics education; to show positive and negative aspects of the impact which computer experiments involved in the process of learning mathematics may have on the formation of the style of pupils' scientific thinking; to call for reasonable and prudent use of computer experiments both in mathematics and mathematics education.

Keywords: experimental mathematics, mathematics education, dynamic geometry software, computer experiment, experimental-theoretical gap

1. Experimental and Theoretical Basis of Mathematics and the Realities of School Education

It is well known that mathematics arose because of practical needs of people. A necessity to *count* different sets of objects led to a concept of *calculation.* A necessity of tax collection led step-by-step to the idea that for *any* sum of money, which had already been collected, one can *add one unit* more. So, the positive integers and arithmetic came to light. Geometry arose from the practice of land measuring and finding the volumes of bodies. Calculus was a kind of a physical and geometrical theory at the very beginning of its existence. To confirm this statement one can think of the method of indivisible continuous by B. Cavalieri (1598 – 1647) (Cavalieri, 1653) or the method of mechanical theorems by Archimedes (287 BC – 212 BC) (Heath, T., 1897).A necessity to analyze gambling gave rise to probability theory. "The book about game of dice" by D. Cardano (1501 – 1576) was one of the first work in this area of mathematics. A necessity to govern by states led to statistics. The book "Political arithmetic" by W. Petty (1623 – 1687) was devoted to statistics as well as to politics.

Such a situation is typical for many areas of mathematics. It is vividly expressed by J. von Neumann: "The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any science which interprets experience on a higher than purely descriptive level" (Von Neumann, 1947). V. Arnold formulated a radical point of view: "Mathematics is an experimental science. It is a part of theoretical physics and a member of the family of natural sciences" (Arnold, V., 2001). We will not represent the whole variety of views on the nature of mathematics, but we state that observation and experiment played an important role in its emergence.

It is natural to understand the role of experimental mathematics in school educations as well as the forms of its use.

Let us analyze how the commutative law of addition is studied at primary school. At first, pupils add two quantities of similar objects in different orders: 2 + 3 is equal to five, and 3 + 2 is also five; 6 + 4 is ten, and 4 + 6 is also equal to ten; etc. After a series of tests, the length of which depends on the specific pedagogical situation, we can formulate the rule:the sum does not depend on the order of addends. It is obvious, that the commutative law of addition is a theoretical fact, but it is not proved because it is not implied from more general statements. Pupils are very young consequently we can neither provide general statement nor prove the commutative law. We convince our pupils by means of a huge number of examples as well as by impossibility to find a counterexample. By the way, pupils have no reasons to look for a counterexample.

Another example of using experimental methods in teaching mathematics is a rule of looking for the unknown addend (Figure 1). We can obtain the rule if we treat Figure 1 in two different ways: firstly, Figure 1 shows how to find the whole length b if we know its parts a and x; secondly, Figure 1 shows, how to find the unknown part x if we know the whole length b and its part a.

$$x + a = b$$
$$x = b - a$$

Figure 1.

One more example is provided by a balance (Figure 2). A balance can be used to obtain a rule of transferring a summand from one side of the equality to another.

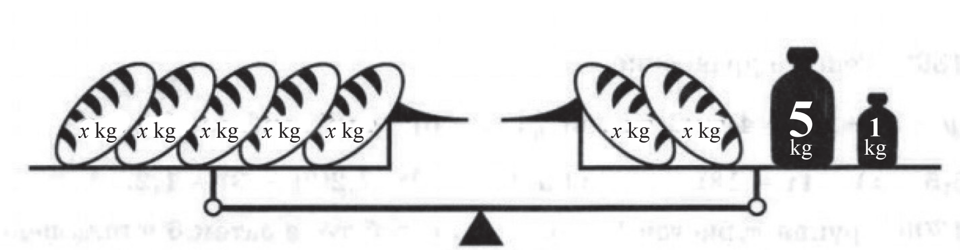

Figure 2.

It is important that such a relationship between experimental and theoretical origins of mathematics is typical for the first years of mathematics education. Here is a list of mathematical fact, which is obtained by means of experiments: 1) the associative law of addition (multiplication); 2) the commutative law of multiplication; 3) the distributive law of multiplication with respect to addition; 4) the rule of adding zero; 5) the rule of multiplication by a unit (zero); 6) the rule of discovering for the unknown minuend (subtrahend, factor, dividend, divisor); 7) the rule of addition (subtraction, multiplication, division) in a column; 8) the rules of arithmetical operations over fractions; 9) the main property of fraction. Of cause, this list is not full, but it allows us to state the following: *pupils from the 1st to 6th grades are mastering mathematics by means of* ***experimental*** *methods mainly.*

Let us remember a well-known idea by J. Bruner: “The schoolboy learning physics *is* a physicist, and it is easier for him to learn physics behave like a physicist than doing something else” (Brunner, J., 1977, p. 14). (Italic by Bruner.)Is natural to assume, that using experimental methods in learning mathematics generates specific skills and a mentality of a mathematician-experimentalist in pupils.

It is interesting that school-student with the mentality of experimentalists go through stern tests because they have to master a theoretical component of mathematics in the 7th and further grades. They have to muster an idea that a statement can be *logically deduced* from the other statements. They meet some new notions such as *"an axiom"* and "*a theorem"*. In connection with theorems, it is necessary to introduce such notions as *"the condition of a theorem", "the conclusion of a theorem", and "the proof of a theorem"*. In connection with proofs, it is necessary to introduce such notion as *"a method of proving"* and to master and collect some methods such as *"a method of identical transformations"* in algebra, *"a method of additional constructions"* in geometry, and a general method *"proof by contradiction"*.

It is natural that skills of a mathematician-experimentalist, which have already been formed, interact with skills of a mathematician-theorist, which are forming right now. At the beginning, this interaction has a character of confrontation. In our opinion, confrontation is based on the inertness of thinking. But it does not matter. Anyway, traditions, "pre-computer" study of mathematics is harmonious because in several years most of students accept the elements of axiomatic method and muster specific geometrical facts obtained within it.

Now let us look at innovative "computer" way of studying mathematics. We consider it a specific way to formulate and to solve problems, to demonstrate statements by means of computer visualizations, to experiment with mathematical objects by means of computers. Opportunities for such an approach provide a whole class of software for educational and scientific purposes. They have general name "dynamic geometry software." We have to mention such well-known products as Cabri-geometre, 1985; The Geometer's Sketchpad, 1989; GeoNext, 1999; GeoGebra, 2002; 1C: Mathematical constructor and other (the year indicates the beginning of its distribution). Today the total number of such products is greater than 50. Their abilities are not limited to geometry. They allow us to support students' learning activities that are relevant to many areas of mathematics: elementary algebra, trigonometry, vector algebra, calculus, graph theory, theory of probability and the mathematical statistics, etc.

Including computers in teaching and learning mathematics generates a number of questions. What is the role of dynamical geometry software in mastering mathematics (geometry, in particular)? If we want students to master both components of mathematics, theoretical and experimental, what is the best way of using dynamical geometry software? Below we will try to answer these questions.

2. Some Framework Conditions for a Use of Dynamical Geometry Software

Dynamical geometry software (DGS) was created to introduce mathematical experiments into a process of teaching geometry. Such experiments were necessary for students to put forward (and further, to verify) hypotheses concerning the properties of geometrical figures. We are happy to say that such a purpose was successfully reached. As an evidence, one can see numerous papers and books published by authors from different countries (for example, (Mariotti, A., 2000), (Ryjik, V., 2000), (Laborde, C., 2001), (Gawlick, Th., 2001), (Shabanova, M. & others, 2013), (Grozdev, S., 2002, 2007, 2014), and many others). At the same time, some negative, unpleasant consequences of using DGS came to light. The authors described them in paper (Shabanova, M. & others, 2014). One of the negative consequences was so important that a special name was given to it. We mean so-called “experimental-theoretical gap”. Its appearance looks as follows: students lose motivation to carry out any deductive reasoning. As a consequence, their ability to do deductive arguments is decreasing as well as their interest to theoretical search, to statement new problems, to logical transformation of problems, etc.

Let us show that an appearance of the experimental-theoretical gap can be prognosticated. Moreover, its appearance is unavoidable, if there are no preventive actions. For the beginning, let us analyze the notion of “proof”.

V. Uspensky, who is a mathematician and a linguist, wrote that the term “proof” was one of the main terms of mathematics. In spite of that, the exact definition of the term does not exist. This fact is not surprising because the term “proof” does not belong to mathematics. (Observe that the term “formal proof” belongs to mathematics.) In fact, it belongs to logics, to linguistics, and to *psychology* (Uspenski, V., 2001, p. 441). An approximate definition looks as follows: “A proof is a persuasive reasoning, which *convinces* us in such a way that we are ready to convince others by means of it”.

Starting with non-mathematical words “psychology” and “convince”, let us consider some classical theorems of geometry and analyze the consequence of using the DGS in process of their study.

Let us assume a computer class equipped with a DGS (to be definite, GeoGebra). It is easy to obtain, that three medians of a triangle pass through a common point. In a computer class this observation can be easily confirmed by means of *hundreds* experiments. By the way, each student can do such experiments *personally.* If some one wants to find a counterexample, it is quite likely that he/she will not be able to do it. As a result, each student and a group as a whole find themselves in a *familiar* situation, i.e., in the situation when they study laws of addition, laws of multiplication, etc. The similarity of different situations activates the existing reasoning of a mathematician-experimentalist. As a result, all students come to a conclusion that their observation is really true. Thus from the psychological viewpoint the statement about medians of a triangle is so convincing that it does not need a deductive proof.

In such a way there appears an experimental-theoretical gap. We repeat that its cause is the following: the skills of an experimentalist (which already exist) interact with skills of a theorist (which are developing at the present moment), more exactly, oppose them at the very beginning.

In order to avoid a negative influence of DGS, teachers should take some special efforts, both at the level of methodology and at the level of everyday work with students.

Methodological presumptions. First of all, teachers have to realize that the experimental and theoretical origins of mathematics are of the same importance. They have to understand that both origins coexist and interact during all periods of mathematical history. They have to realize that a mathematical way of reasoning consists of both origins. Secondly, teachers have to take into account that both origins of mathematics interact dialectically when students master a mathematical way of reasoning. In particular, this means that if one applies DGS as an instrument of teaching, it is reasonable to follow with so-called *principle of accumulative evolution.* Let us look at it in more detail.

For the first time, principle of accumulative evolution was formulated in paper (Yastrebov, A. & I. Zavyalova, 2012) with regard to electronic book of problems. As this topic is far from the subject of the present paper, let us reformulate the principle: *a use of DGS should A) preserve all of the achievements of a traditional teaching, which existed "before computers"; B) eliminate some defects of a traditional teaching; C) provide the teacher with some additional pedagogical instruments, which are generated by capabilities of computers.*

In order to reach the above mentioned purposes, teachers have to use some special *methods.* These methods can be treated "methods of teaching students" as well as "methods of self-teaching for teachers". Let us describe some of them.

Prevention of illusions. Let us consider some mathematical problems and analyze them from the viewpoints of mathematics, didactics of mathematics, and studying of GeoGebra as a whole. The first problem could be given at the very beginning of study GeoGebra.

Problem 1. Put two points, O (0, 0) and A = (0.000000000000001,0), into graphic field of GeoGebra. Find the distance between these points.

Analysis. Firstly, we have to establish an accuracy of approximation up to 15 decimal digits. Secondly, we can put two arbitrary points into the graphic field of GeoGebra and then substitute their co-ordinates in the panel on objects for the required ones. Thirdly, we can indicate the points in the panel of objects by the instrument "Distance".

As a result, we obtain that OA = 0.000000000000001 = 10^{-15}cm. From the formal point of view, everything is OK. On the other hand, the distance of 10^{-15} cm is 1000 times less than the linear size of the atomic nucleus! There is no instrument for measuring such distances. They will not appear in the nearest future. Thus, GeoGebra does not deal with traditional geometry. It deals with geometry of arithmetical plane $\boldsymbol{R}^2$, represents its objects (pairs of numbers, linear equations, etc.) as objects of Euclidean geometry, and shows them in a discrete screen.

Thus, *an unskilled user does not obtain an illusion that GeoGebra is a purely geometrical instrument.* Of course, an advanced user knows that the arithmetical plane $\boldsymbol{R}^2$ is an isomorphic model of Euclidean geometry, but, in general, a high-school student is not an advanced user.

It is reasonable to provide students the next problem *before* they are familiar with the Pythagorean theorem.

Problem 2. 1) Find the diagonal x of a square with the side of 1 cm, as well as the diagonal y of a rectangle with the sides of 3 cm and 4 cm. 2) Two squares with the sides of 1 cm are cut along their diagonals. Put four triangles into a new square with the side x. Find the area of the new square in two different ways. 3) Two rectangles with the sides of 3 cm and 4 cm are cut along their diagonals. Put four triangles into a new square with the side y. Find the area of the new square in two different ways.

Analysis. 1) One can find the length of a segment either by an appropriate formula or by measurement. As the formula is unknown, a student has to use an instrument of GeoGebra. If the setup is standard, he/she will obtain that x = 1,41 cm and y = 5 cm.

At first sight, the problem has been solved, but our further activity reveals some irregularities. If we choose an approximation up to 3 decimal digits, we obtain that x = 1.414 cm. If we choose an approximation up to 4 decimal digits, we obtain that x = 1.4142 cm. It is interesting that y = 5 cm *independently of* the way of approximation! Why does GeoGebra give different answers in two similar situations? Why is the first answer approximate and the second answer exact? In order to answer these questions, we have to work with items 2 and 3 of our problem.

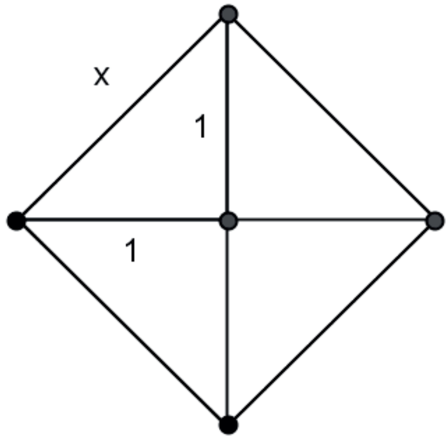

Figure 3.

2) A square in Figure 3 consists of four non-intersecting rectangular triangles with legs of 1 cm; consequently, its area can be found in two different ways: $S = x^2 = 4 \cdot \frac{1}{2} \cdot 1 \cdot 1$.. It means that the diagonal of the initial square satisfies an equality $x^2 = 2$. Let us note that none of our results (1.41, 1.414, and 1.4142) satisfies the latter equality. Thus, school-students meet the same problem as Pythagoras met ages ago! Unlike of the ancient problem, GeoGebra can "help" students because its instrument CAS will provide two solutions, $\sqrt{2}$ and $-\sqrt{2}$. Further students will understand the sense of the new symbol $\sqrt{2}$.

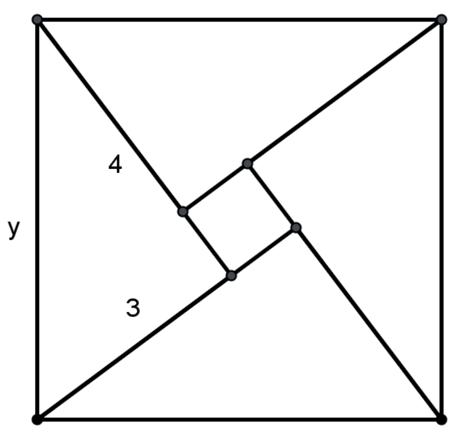

Figure 4.

3) A square in Figure 4 consists of four non-intersecting rectangular triangles with legs of 3 cm and 4 cm and a square with a side of 1 cm. Consequently, its area can be found in two different ways: $S = y^2 = 4 \cdot \frac{1}{2} \cdot 3 \cdot 4 + 1^2$. Consequently, $y^2 = 25$ and $y = 5$. This result explains why the answer $y = 5$ did not depend on the way of approximation.

Let us mention, that Figure 4 can be found in Egyptian tombs erected 2000 B.C.

The analysis of problem 2 leads us to two different conclusions. Firstly, we can use the items of problem 2 as a preliminary study of Pythagorean Theorem as well as a notion of an irrational number. Moreover, they can be used in the process of study of Pythagorean Theorem, as it is provided in paper (Rojkova, S. & A. Yastrebov, 2010). Secondly, students can independently come to the conclusion that GeoGebra's "measurements" are approximate and need analytical user's understanding. Thus, *an unskilled user doesn't obtain an illusion that GeoGebra provides him/her absolutely trusted results.*

Education of an experimentalist's world-view. The following statement is generally recognized: mathematics education has to form a scientific world-view. Let us demonstrate some simple problems, which help to form a world-view of a mathematician-experimentalist.

Problem 3. In the graphic field of GeoGebra you can see a geometrical figure and a question. Answer the questions concerning properties of the figure. To be exact, there is one point in the graphic field. The questions are: "How many points are there in the picture? What are their co-ordinates?" Such a field is provided to students *four times.*

Analysis. A situation looks paradoxical, because a point is *unique*, but a question is formulated with respect to *several* points. If a student turns on a panel of objects, the situation becomes even more paradoxical, because the panel of objects has four different appearances in four cases, as it is shown in the following table:

1	2	3	4
A (1, 2)	*A* (1, 2) *B* (1.01, 2) *C* (1, 2.02)	*A* (1, 2) *B* (1, 2) *C* (1, 2)	*A* (1, 2) *B* (1, 2) *C* (1, 2)

In the first case there is one point. *In the second case* there are three points, which are placed very close to each other. They are so close that their images coincide and look as a single point. *The third case* looks like the first one, but it is not clear why a single point has three different names. A student can solve this mini-problem if he/she chooses an approximation up to 3 decimal digits. In such a case *A* (1, 2), *B* (1.001, 2), *C* (1, 2.001).

One can see that there are three points, which are closer to each other than in case 2. At first sight, *the fourth case* coincides with the third one, but any change of approximation does not change the co-ordinates of the points. As a result, a student comes to a conclusion that a single point can have different names. An explanation looks as follows: three different points move on the plane independently of each other and *coincide* at a special moment of time; this moment is fixed in the last picture.

This problem, as well as other problems of such a type, leads students to a belief, which is ***the basic for a world-view of a mathematician-experimentalist:*** *any observation needs a theoretical understanding.* Thus, experimental-theoretical dualism of mathematics comes to light. It is reasonable to consider this kind of dualism together with common property of mathematics, i.e. its empirical-theoretical dualism (Yastrebov, A., 2001).

Figures as a stimulus for research. It is natural to think that drawings of ancient mathematicians motivated them to further research. A cause of such a statement looks as follows. Figures were drawn on sheets of papyrus. They were not large, and their surface was rough. Drawings were made by a sharpened stalk of reed, which made thick lines. As a result, drawings had some properties which were in contradiction to each other. On the one hand, figures helped to formulate a hypothesis concerning properties of geometrical objects under consideration. On the other hand, figures could not convince mathematicians that their hypothesis was true. Under such conditions, a deductive reasoning was a unique method to establish the truth.

Drawings by GeoGebra are much more exact than drawings of ancient mathematicians. Besides, they are dynamic. The latter property allows us not only to formulate a hypothesis, but to check it in many particular cases. It looks paradoxical but exactness of drawings reduces their stimulating influence. Under such conditions, teachers should construct plenty of situations when drawings by GeoGebra have the same stimulating influence as drawings of ancient mathematicians. Let us show that *different drawings* stimulate further research in *different extents*.

Problem 4. 1) Draw three medians of a triangle and formulate a hypothesis about their mutual disposition. 2) Measure three angles of a triangle, add them and formulate a hypothesis about the sum.

Analysis. 1) *A beautiful* result is evident: three medians have a common point. If we find their pairwise intersections, we obtain that three points have the same co-ordinates, therefore, they coincide. If we increase accuracy of approximation up to 5 decimal digits, we can confirm our statement. At the same time, we can obtain a *paradoxical* result, if we set up maximal accuracy of approximation, i.e., 15 decimal digits. In such a case there exist triangles with *three different points of the pair wise intersections*. It is evident that a contradiction between “beautiful results” and “paradoxical results” can be solved by means of the deductive method. We must mention that teachers have to make some special efforts to obtain a paradoxical result.

If we measure the angles of a triangle and add them, we obtain an absolutely stable, invariant result of 180^{o}, which does not depend on the accuracy of approximation. The hypothesis generated by our result looks well-grounded and does not stimulate further research.

It goes without saying that the above-mentioned problems are individual examples. The list of these problems should be increased and expanded to different areas of geometry. Besides, it is necessary to verify pedagogical efficiency of problems of such a kind. In spite of this, the authors are ready to provide some recommendations concerning education of a mathematician-experimentalist as well as elimination of experimental-theoretical gap.

The first recommendation is a general presumption: if we teach geometry, we should form a harmonic attitude to DGS, i.e. we should form *simultaneously* trust and distrust to “hints” of DGS.

The second recommendation is more categorical: we should begin a usage of DGS from such situations, when DGS helps students to formulate a hypothesis, but does not convince them of its truth.

3. Soft Manifesto of Experimental Mathematics

According to the memoirs of academician N. N. Krasilnikov, the term “experimental mathematics” was said for the first time in 1969, when the Academy of Science of the USSR established their Ural Department. Nowadays, we can see different attitudes to this term. Some scientists think that the term is no more than a new brand (McEvoy, M., 2013). Others believe that it is a name of a new area of mathematics. For example, “A manifesto of experimental mathematics” was published by a group consisting of specialists from the Russian Peoples Friendship University and the Institute of Oceanology named after P. P. Shirshov (Shamin, R., 2008).

It is interesting that a boundary of experimental mathematics is rigidly determined in “A manifesto…”. Besides this Manifesto does not mention a usage of experimental methods in mathematics education. Meanwhile, in order to understand an evolution of the role of the experimental component of mathematics, it would be natural to join efforts of mathematicians and mathematics educators.

Previously we mentioned the usage of experimental mathematics in education, i.e., a phenomenon, which is new for didactics of mathematics. As usual, this phenomenon has supporters and opponents, enthusiasts and skeptics. Both sensible supporters and sincere skeptics realize that experimental mathematics has some substantial, though not immense, achievements. They realize that its usage in education is fruitful, but meets a number of difficulties. That is why we do believe that mathematicians and educators need a kind of a "soft manifesto", i.e., a text, which is admitted by many specialists. Below we make an attempt to write such a text.

Manifesto

It is well known that mathematics, like any science, has a dual nature. On the one hand, it represents an activity of getting new knowledge in its specific field, and on the other hand, it is a sum of knowledge gained at a given moment. That is why in teaching mathematics at all levels, it is necessary that students and pupils should learn mathematical facts and master research skills in mathematics, and both must go on simultaneously and equally. In particular, the process of teaching should include mathematical experiments, because mathematics in the making was an experimental science, and it has retained both bases, theoretical and experimental.

The researcher's activities with the objects of the material world or their ideal images will be assigned to the field of ***experimental mathematics,*** *if its results are hypotheses about the properties of mathematical objects and/or mathematical pre-concepts or concepts.*

Different nations had various tools of mathematical experiments at different times such as dice, playing cards and coins; square sheets of origami paper; a real pair of compasses, a real ruler, real papyrus and paper (later); an ideal ruler and an ideal compass; a protractor, a double-sided ruler and a stencil of the right angle; a computer, etc.

The computer plays a special role among the tools used in mathematical experiments. Its role in the experiments is so crucial; the obtained results are so interesting and varied, that we can speak about the *emergence* of experimental mathematics as a special field of mathematics and of the *identification* of the mathematical experiment with the computer experiment. Apparently, the words "emergence" and "identification" are kinds of hyperbole and in this sense are not accurate, but they reflect a new reality i.e. a sharp increase in the role of the experimental component of mathematics.

It should be mentioned that the mathematical experiments have been actively used in education. Digital educational resources give an opportunity to organize mathematical experiment in the process of teaching and learning. This fact has given rise to strong positive effects on the one hand, and has revealed serious risks, on the other hand.

Mathematicians and teachers of mathematics have a noble goal of learning how to use experimental methods for the development of mathematics and pedagogy of mathematics.

REFERENCES

Cavalieri, B. (1653) *Geometriain divisibilibus continuorum nova quadam ratione promota.* 2nd Ed., Bologna: Typo graphed Duiis.

Heath, L. (1897) *The Works Of Archimedes*, Eds. C.J. Clay and sons, Cambridge University Press Warehouse, 524 pages.

Von Neumann, J. (1947) *The Mathematician. Works of the Mind*, Vol. I no. 1 University of Chicago Press, Chicago, P. 180 – 196.

Arnold,V. (2000) *"Rigid" and "soft" mathematical models.* M.: Mccme. – 32 pages.

Bruner, J. (1977) *The Process of Education*, Harvard University Press. – 128 pages.

Mariotti, A. (2000) Introduction to proof: The mediation of a dynamic software environment, *Educational Studies in Mathematics*, 44(1). – P. 25 – 53.

Ryjik,V. (2000) Geometry and computer, *Computer instruments in education*. № 6. – P. 7 – 11.

Laborde, C. (2001) Integration of technology in the design of geometry tasks with cabri-geometry, *International Journal of Computers for Mathematical Learning,* 6(3). – P. 283 – 317.

Gawlick, Th. (2005) *Connecting Arguments to Actions – Dynamic Geometry as Means for the Attainment of Higher van Hiele Levels*. ZDM, 37(5). – P. 361 – 370.

Shavanova, M. and others. (2013) *Teaching mathematics with using GeoGebra's abilities: collective monograph*. M.: Pero. – 128 pages.

Grozdev, S. (2002) Mathematical modeling of educational process, *Journal of Theoretical and Applied Mechanics*, 32, 1, 2002, 85 – 90.

Grozdev, S. (2007) *For High Achievements in Mathematics, The Bulgarian Experience (Theory and Practice)*. ADE, Sofia, 2007. (ISBN 978-954-92139-1-1), 295 pages.

Sergeeva, T, Shabanova, M., Grozdev, S. (2014) *Foundation of dynamic geometry.* –M.: АСОУ. – 160 pages.

Shabanova, M., Yastrebov, A., Bezumova, O., Kotova, S., Pavlova, M. (2014) Experimental mathematics and mathematics education, *International Multidisciplinary scientific conferences on social sciences and arts: psychology and psychiatry, sociology and healthcare, education*. Conference proceedings – Volume III – Sofia, Bulgaria. – P. 309 – 320.

Uspenski, V. (2011) *Apology of mathematics.* – Sent-Petersburg: Amphora. – 554 pages.

Yastrebov, A., Zavyalova, I. (2012) Principle of accumulation education as a base of composing of electronic book of problem, *Problems of teaching mathematics at schools and universities in a context of new educational standards: Synopsis of reports given at XXXI All-Russia Seminar of university professors, devoted to 25 anniversary of the Seminar (26–29 of September 2012, Tobolsk)*. – Tobolsk: TGCPA. – P. 208 – 209.

Rojkova, S., Yastrebov, A. (2010) Methods of areas and its consequences of pedagogical and mathematical nature. – Mathematics, Physics, Economy, Technology and their teaching: *Proceeding of conference "Ushinsky Readings" of the Departments of Mathematics and Physics.* Part II. – Yaroslavl, YSPU. – P. 79 – 86.

Yastrebov, A. (2001) Dualistic properties of mathematics and their reflection in teaching, *Yaroslavl Pedagogical Gerald*.– № 1. – P. 48 – 53.

McEvoy, M. (2013) *Experimental mathematics, computer sand the apriori*, Philosophical research on-line, Synthese 190, Netherlands, P. 397 – 412.

Shamin, R. (2008) *A manifesto of experimental mathematics.* – URL: http://xmath.ru

✉ **Prof. Alexander Yastrebov, DSc.**
Yaroslavl State Pedagogical University
108, Respublikanskaya Street
Yaroslavl, Russia
E-mail: a.yastrebov@yspu.org

✉ **Prof. Maria Shabanova, DSc.**
Institute of Mathematics, Information and Space Technologies
17, Naberezhnaja Severnoj Dvini Street
Arkhangelsk, Russia
E-mail: m.shabanova@narfu.ru