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DYNAMICS OF A NEW CLASS OF OSCILLATORS: MELNIKOV'S APPROACH, POSSIBLE APPLICATION TO ANTENNA ARRAY THEORY

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Abstract. In this paper, we propose a new class of extended oscillators. Some investigations based on the Melnikov's approach are applied for identifying some chaotic possibilities. We demonstrate also some specialized modules for investigating the dynamics of these oscillators. One possible application that Melnikov functions may find in the modeling and synthesis of radiating antenna patterns is also discussed. This will be included as an integral part of a planned much more general Web-based application for scientific computing.

Keywords: escape oscillator; Melnikov's method; hypothetical Melnikov antenna array

1. Introduction

The author of (Sanjuan 1999) considers the equation of motion for the sinusoidally driven escape oscillator including nonlinear damping terms as a power series on the velocity reads

$$\ddot{x} + \sum_{p=1}^{n} \beta \dot{x}|\dot{x}|^{p-1} + x - x^2 = F \sin \omega t, \tag{1}$$

where β is the damping level, p is the damping exponent, and F and ω the forcing amplitude and the frequency of the external perturbation, respectively. In particular, the following model is considered

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 - \beta y|y|^{p-1} + F \sin \omega t, \end{cases} \tag{2}$$

where for simplicity only a single damping term proportional to the p^{th} power of the velocity is taken. For other results, see (Soltman & Thompson 1992), (Fangnon et al. 2020), (Bikdash et al. 1994), (Ravindra & Mallik 1984, 1999), (Sanjuan 1996), (Guckenheimer & Holms 1983), (Perko 1991), (Gavrilov & Iliev 2015), (Holmes & Marsden 1981, 1982). In this paper, we suggest a new class of modified oscillators and study their dynamics. The first important task we investigate is the possible chaotic behavior of our model. To do this we use the method proposed by Melnikov in (Melnikov 1963). Several simulations are composed. We demonstrate also some specialized modules for investigating the dynamics of these hypothetical escape oscillators. The derived results can be used as an integral part of a much more general application for scientific computing – for some details see (Kyurkchiev & Zaevski 2023), (Kyurkchiev et al. 2023), (Kyurkchiev & Iliev 2022), (Kyurkchiev et al. 2022), (Vasileva et al. 2023), (Golev et al. 2024), (Kyurkchiev et al 2024), (Kyurkchiev et al. 2024a). One possible application that Melnikov functions may find in the modeling of radiating antenna patterns is discussed in Section 4.1.

2. The model

We consider the following new class of extended escape oscillators

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 - \epsilon \left(A \displaystyle\sum_{p=1}^{n} y|y|^{p-1} + \sum_{j=1}^{N} g_j \sin(j\omega t) \right), \end{cases} \tag{3}$$

where $0 \leq \epsilon < 1$, A is the damping level, $p \geq 1$ is the damping exponent, and N is integer. In particular, we consider the following model

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -x + x^2 - \epsilon \left(Ay|y|^{p-1} + \displaystyle\sum_{j=1}^{N} g_j \sin(j\omega t) \right). \end{cases} \tag{4}$$

For $\epsilon = 0$, the resulting Hamiltonian of the system (4) is $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$.

The homoclinic orbit is given by (see fig.1)

$$x_0(t) = 1 - \frac{3}{1 + cosh\, t}, y_0(t) = \frac{3sinh\, t}{(1 + cosh\, t)^2}. \tag{5}$$

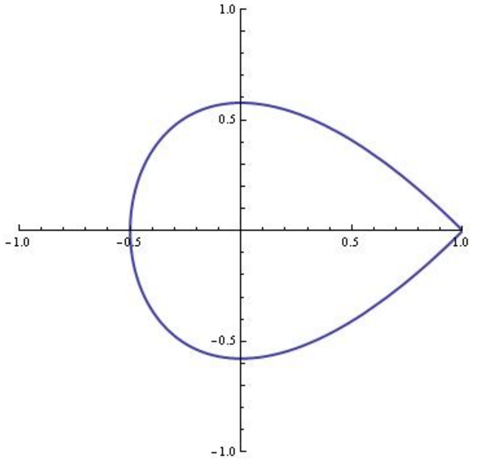

Figure 1. The homoclinic orbit (Fangnon et al. 2020)

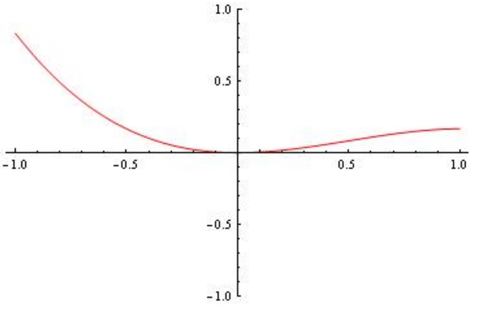

Figure 2. Potential energy $V(x)$

The potential energy $V(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ is shown in fig. 2.

3. Results in the light of the Melnikov's approach

The Melnikov function gives a measure of the leading order distance between the stable and unstable manifolds when $\epsilon \neq 0$ and can be used to tell where the stable and unstable manifolds intersect transversely. By definition, the homoclinic integral of Melnikov is given by

$$M(t_0) = \int_{-\infty}^{\infty} y_0(t) \left(A \sum_{p=1}^{n} y_0(t)|y_0(t)|^{p-1} + \sum_{j=1}^{N} g_j \sin(j\omega(t + t_0)) \right) dt, \quad (6)$$

where the functions $x_0(t)$ and $y_0(t)$ are defined by equations (4).

From a numerical point of view, the task of finding the roots of $M(t_0) = 0$ is more interesting given that the parameters appearing in the proposed differential model are subject to a number of restrictions of a physical and practical nature.

3.1. The case $p = 2$ *and* $N = 1$ *(model (4))*

We can prove the following proposition when $p = 2$ and $N = 1$.

Proposition 1. *If* $p = 2$ *and* $N = 1$*, then the roots of Melnikov function* $M(t_0)$ *are given as solutions of the nonlinear equation:*

$$M(t_0) = \frac{9}{16}A + \frac{3}{16}e^{-it_0\omega}\big(16g_1\omega + 16e^{2it_0\omega}g_1\omega -\\ -8ie^{2it_0\omega}g_1\omega^2\mathcal{H}[-\tfrac{1}{2} - \tfrac{i\omega}{2}] +\\ +8ie^{2it_0\omega}g_1\omega^2\mathcal{H}[-\tfrac{i\omega}{2}] - 8ig_1\omega^2\mathcal{H}[\tfrac{i\omega}{2}] +\\ +8ig_1\omega^2\mathcal{H}[\tfrac{i(i+\omega)}{2}] - 8ig_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] +\\ +8ig_1\omega^2\mathcal{P}[0, 1 - \tfrac{i\omega}{2}] + 8ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] -\\ -8ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, 1 + \tfrac{i\omega}{2}]\big) = 0 \tag{7}$$

For example, for $A = 1.39627$, $\omega = 0.2$, $g_1 = 0.69$ the root of the nonlinear equation (7) (in interval $(0, 30)$) is $t_0 \approx 15.708$ with multiplicity two (see fig. 3).

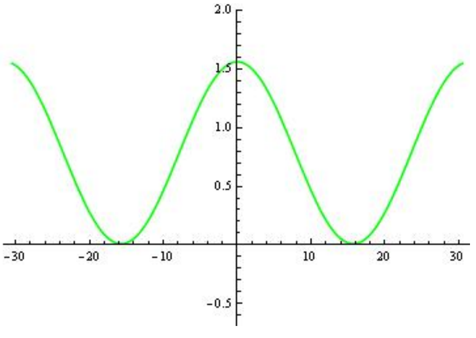

Figure 3. The nonlinear equation (7)

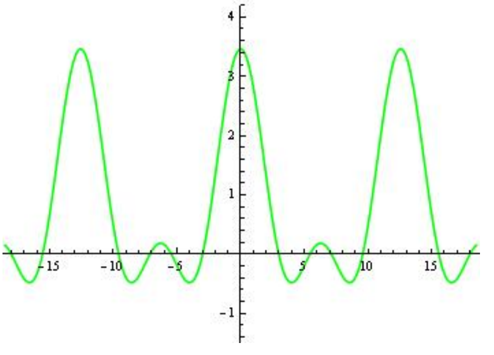

Figure 4. The nonlinear equation (8)

3.2. The case $p = 2$ and $N = 2$ (model (4))

We can prove the following proposition when $p = 2$ and $N = 2$.

Proposition 2. *If $p = 2$ and $N = 2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation:*

$$M(t_0) = \frac{9}{16}A + \frac{3}{16}e^{-2it_0\omega}\big(16e^{it_0\omega}g_1\omega + 16e^{3it_0\omega}g_1\omega\\ +32g_2\omega + 32e^{4it_0\omega}g_2\omega\\ -8ie^{it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] - 8ie^{3it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}]\\ +8ie^{it_0\omega}g_1\omega^2\mathcal{P}[0, 1 - \tfrac{i\omega}{2}] + 8ie^{3it_0\omega}g_1\omega^2\mathcal{P}[0, 1 - \tfrac{i\omega}{2}]\\ +8ie^{it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] + 8ie^{3it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}]\\ -8ie^{it_0\omega}g_1\omega^2\mathcal{P}[0, 1 + \tfrac{i\omega}{2}] - 8ie^{3it_0\omega}g_1\omega^2\mathcal{P}[0, 1 + \tfrac{i\omega}{2}]\\ -32ig_2\omega^2\mathcal{P}[0, \tfrac{1}{2} - i\omega] - 32ie^{4it_0\omega}g_2\omega^2\mathcal{P}[0, \tfrac{1}{2} - i\omega]\\ -32ig_2\omega^2\mathcal{P}[0, 1 - i\omega] + 32ie^{4it_0\omega}g_2\omega^2\mathcal{P}[0, 1 - i\omega]\\ +32ig_2\omega^2\mathcal{P}[0, \tfrac{1}{2} + i\omega] + 32ie^{4it_0\omega}g_2\omega^2\mathcal{P}[0, \tfrac{1}{2} + i\omega]\\ -32ig_2\omega^2\mathcal{P}[0, 1 + i\omega] - 32ie^{4it_0\omega}g_2\omega^2\mathcal{P}[0, 1 + i\omega]\big) = 0. \tag{8}$$

For example, for $A = 1.5$, $\omega = 0.5$, $g_1 = 0.8$, $g_2 = 0.6$ the roots of the nonlinear equation (8) (in the interval $(0, 6)$) are $t_0 \approx 2.98971$ and $t_0 \approx 5.43912$ (see fig. 4).

3.3. The case $p = 2$ ***and*** $N = 3$ ***(model (4))***

Proposition 3. *If* $p = 2$ *and* $N = 3$*, then the roots of Melnikov function* $M(t_0)$ *are given as solutions of the equation:*

$$
\begin{aligned}
M(t_0) \;=\;& \tfrac{1}{60}e^{-3it_0\omega}\big(32\sqrt{2}Ae^{3it_0\omega} + 15\sqrt{2}e^{2it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{i\omega}{4}] \\
&+15\sqrt{2}e^{4it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{i\omega}{4}] - 15\sqrt{2}e^{2it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{i\omega}{4}] \\
&-15\sqrt{2}e^{4it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{i\omega}{4}] + 15\sqrt{2}e^{2it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{i\omega}{4}] \\
&+15\sqrt{2}e^{4it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{i\omega}{4}] - 15\sqrt{2}e^{2it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{i\omega}{4}] \\
&-15\sqrt{2}e^{4it_0\omega}g_1\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{i\omega}{4}] + 30\sqrt{2}e^{it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{i\omega}{2}] \\
&+30\sqrt{2}e^{5it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{i\omega}{2}] - 30\sqrt{2}e^{it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{i\omega}{2}] \\
&-30\sqrt{2}e^{5it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{i\omega}{2}] + 30\sqrt{2}e^{it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{i\omega}{2}] \\
&+30\sqrt{2}e^{5it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{i\omega}{2}] - 30\sqrt{2}e^{it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{i\omega}{2}] \\
&-30\sqrt{2}e^{5it_0\omega}g_2\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{i\omega}{2}] + 45\sqrt{2}g_3\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{3i\omega}{4}] \\
&+45\sqrt{2}e^{6it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{4} - \tfrac{3i\omega}{4}] - 45\sqrt{2}g_3\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{3i\omega}{4}] \\
&-45\sqrt{2}e^{6it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{3}{4} - \tfrac{3i\omega}{4}] + 45\sqrt{2}g_3\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{3i\omega}{4}] \\
&+45\sqrt{2}e^{6it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{4} + \tfrac{3i\omega}{4}] - 45\sqrt{2}g_3\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{3i\omega}{4}] \\
&-45\sqrt{2}e^{6it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{3}{4} + \tfrac{3i\omega}{4}]\big) = 0.
\end{aligned}
\tag{9}
$$

Remark. The case $p = 2$, $N = 3$ is considered in (Kyurkchiev et al. 2024) only as an illustration of the difficulties that the user encounters in the study of such dynamic models using CAS. We include Proposition 3 here only for completeness.

3.4. The case $p = 4$ ***and*** $N = 1$ ***(model (4))***

We can prove the following proposition when $p = 4$ and $N = 1$.

Proposition 4. *If* $p = 4$ *and* $N = 1$*, then the roots of Melnikov function* $M(t_0)$ *are given as solutions of the nonlinear equation:*

$$
\begin{aligned}
M(t_0) \;=\;& \tfrac{81}{560}A + \tfrac{3}{560}e^{-it_0\omega}\big(560g_1\omega + 560e^{2it_0\omega}g_1\omega \\
&-280ie^{2it_0\omega}g_1\omega^2\mathcal{H}[-\tfrac{1}{2} - \tfrac{i\omega}{2}] \\
&+280ie^{2it_0\omega}g_1\omega^2\mathcal{H}[-\tfrac{i\omega}{2}] - 280ig_1\omega^2\mathcal{H}[\tfrac{i\omega}{2}] \\
&+280ig_1\omega^2\mathcal{H}[\tfrac{i(i+\omega)}{2}] - 280ig_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] \\
&+280ig_1\omega^2\mathcal{P}[0, 1 - \tfrac{i\omega}{2}] + 280ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] \\
&-280ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, 1 + \tfrac{i\omega}{2}]\big) = 0
\end{aligned}
\tag{10}
$$

3.5. The case $p = 4$ ***and*** $N = 2$ ***(model (4)).***

We can prove the following proposition when $p = 4$ and $N = 2$.

Proposition 5. *If $p = 4$ and $N = 2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation:*

$$\begin{aligned} M(t_0) \;&=\; \tfrac{81}{560}A + 6g_1\pi\omega^2\cos(t_0\omega)csch\,(\pi\omega) \\ &\quad +24g_2\pi\omega^2\cos(2t_0\omega)csch\,(2\pi\omega) = 0. \end{aligned} \tag{11}$$

3.6. The case $p = 4$ and $N = 3$ (model (4)).

We can prove the following proposition when $p = 4$ and $N = 3$.

Proposition 6. *If $p = 4$ and $N = 4$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation:*

$$\begin{aligned} &M(t_0) = \tfrac{3}{35840}e^{-3it_0\omega}\big(1728Ae^{3it_0\omega} + 17920ie^{2it_0\omega}g_3 \\ &-777433545iAe^{3it_0\omega}\pi - 17920e^{2it_0\omega}g_3\omega \\ &-17920e^{4it_0\omega}g_3\omega + 17920ie^{2it_0\omega}g_3\omega\pi csch\,(\pi\omega) \\ &-17920ie^{4it_0\omega}g_3\omega\pi csch\,(\pi\omega) - 17920ie^{2it_0\omega}g_3\omega^2\pi csch\,(\pi\omega) \\ &-17920ie^{4it_0\omega}g_3\omega^2\pi csch\,(\pi\omega) + 143360e^{it_0\omega}g_2\omega^2\pi csch\,(2\pi\omega) \\ &+143360e^{5it_0\omega}g_2\omega^2\pi csch\,(2\pi\omega) + 322560g_3\omega^2\pi csch\,(3\pi\omega) \\ &+322560e^{6it_0\omega}g_3\omega^2\pi csch\,(3\pi\omega) - 8960e^{2it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] \\ &+8960e^{4it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] - 17920ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] \\ &-17920ie^{4it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] - 8960ie^{2it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] \\ &-8960ie^{4it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{1}{2} - \tfrac{i\omega}{2}] + 17920ie^{4it_0\omega}g_1\omega^2\mathcal{P}[0, 1 - \tfrac{i\omega}{2}] \\ &+8960e^{2it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] - 8960e^{4it_0\omega}g_3\omega\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] \\ &+17920ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] + 17920ie^{4it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] \\ &+8960ie^{2it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] + 8960ie^{4it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{1}{2} + \tfrac{i\omega}{2}] \\ &-17920ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, 1 + \tfrac{i\omega}{2}] + 8960e^{2it_0\omega}g_3\omega\mathcal{P}[0, -\tfrac{i\omega}{2}] \\ &-8960e^{4it_0\omega}g_3\omega\mathcal{P}[0, -\tfrac{i\omega}{2}] + 17920ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0, -\tfrac{i\omega}{2}] \\ &+8960e^{2it_0\omega}g_3\omega^2\mathcal{P}[0, -\tfrac{i\omega}{2}] + 8960ie^{4it_0\omega}g_3\omega^2\mathcal{P}[0, -\tfrac{i\omega}{2}] \\ &-17920ie^{4it_0\omega}g_1\omega^2\mathcal{P}[0, \tfrac{i\omega}{2}] - 8960ie^{2it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{i\omega}{2}] \\ &-8960ie^{4it_0\omega}g_3\omega^2\mathcal{P}[0, \tfrac{i\omega}{2}]\big) = 0. \end{aligned} \tag{12}$$

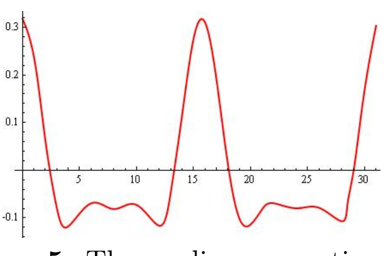

Figure 5. The nonlinear equation (12)

For example, for $A = 10^{-11}$, $\omega = 0.4$, $g_1 = 0.08$, $g_2 = 0.06$, $g_3 = 0.04$ (see fig. 5).

4. Remarks. Open problems

From Propositions 1–6, the reader may formulate the Melnikov's criterion for the appearance of the intersection between the perturbed and unperturbed separatrices. The reader can consider the corresponding approximation problem for arbitrarily chosen n and N (see the general model (3)). For example, we can prove the following proposition when $n = 2$ and $N = 1$ (model (3)).

Proposition 7. *If $n = 2$ and $N = 1$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the nonlinear equation:*

$$\begin{aligned} M(t_0) = &\tfrac{3}{640}e^{-it_0\omega}\big(376Ae^{it_0\omega} - 70575iAe^{it_0\omega}\pi + 640g_1\omega \\ &+640e^{2it_0\omega}g_1\omega - 320ig_1\omega^2\mathcal{P}[0,\tfrac{1}{2}-\tfrac{i\omega}{2}] \\ &-320ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0,\tfrac{1}{2}-\tfrac{i\omega}{2}] + 320ig_1\omega^2\mathcal{P}[0,1-\tfrac{i\omega}{2}] \\ &+320ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0,1-\tfrac{i\omega}{2}] + 320ig_1\omega^2\mathcal{P}[0,\tfrac{1}{2}+\tfrac{i\omega}{2}] \\ &+320ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0,\tfrac{1}{2}+\tfrac{i\omega}{2}] - 320ig_1\omega^2\mathcal{P}[0,1+\tfrac{i\omega}{2}] \\ &-320ie^{2it_0\omega}g_1\omega^2\mathcal{P}[0,1+\tfrac{i\omega}{2}]\big) = 0. \end{aligned} \qquad (13)$$

For p and N (sufficiently large numbers), some difficulties (of a user nature) are encountered when calculating the Melnikov integrals using known Computer Algebraic Systems. The Melnikov function $M(t_0)$ can be evaluated via the method of residuals. Modern computer algebraic systems for scientific calculations provide this opportunity for the users. The upgrade of the Web Application planned by us foresees the use of an algorithm (hidden to the user) to define the limit of the type $|Im(\omega)| \leq Const$. For example, see Proposition 2, we calculate $M(t_0)$ but with a pre-set limit by us: $|Im(\omega)| \leq 1$, for the Proposition 4 – with a pre-set limit by us: $Im(\omega) \leq -\frac{1}{2}$ and for the Proposition 7 – with a pre-set limit by us: $Im(\omega) \leq -1$.

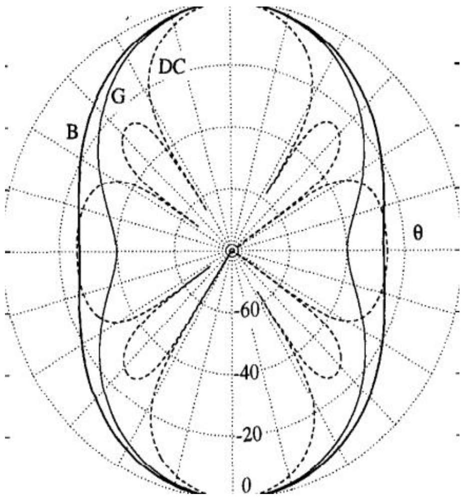

Figure 6. Comparisons between B-binomial, G-present theory by Soltis and DC-Dolph-Chebyshev arrays (Soltis 1993)

Let us consider now one possible application that Melnikov functions may find in the modeling and synthesis of radiating antenna patterns, focusing on $M(t)$. We define the hypothetical normalized antenna factor as follows:

$$M^*(\theta) = \frac{1}{D}|M(K\cos\theta + k_1)|; K = kd;\ k = \frac{2\pi}{\lambda},$$

where θ is the azimuth angle, λ is the wave length, d is the distance between emitters and k_1 is the phase difference.

Following the classical binomial, Dolph-Chebyshev, Gegenbauer-like, Jacoby-like, Soltis antenna array, we will call $M^*(\theta)$ the *Melnikov antenna array.*

Comparisons between B-binomial, G-present theory by Soltis and DC-Dolph-Chebyshev arrays are depicted on fig. 6. For more details see (Soltis 1993).

For fixed $A = 0.0001$; $\omega = 0.2$; $g_1 = 0.008$; $g_2 = 0.006$; $K = 5.5$; $k_1 = 0.01$ (from Proposition 2, see fig. 7). For fixed $A = 0.007$; $\omega = 0.32$; $g_1 = 0.45$; $g_2 = 0.35$; $K = 9.9$; $k_1 = 0.001$ (from Proposition 5, see fig. 8).

Of course, after serious consideration by specialists working in this scientific direction, the hypothetical Melnikov array proposed by us can be seen as a supplement to the Array Antenna Theory.

5. Some simulations

Here we will focus on some interesting simulations

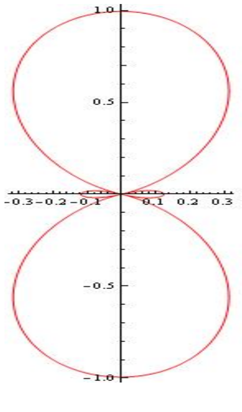

Figure 7. A typical Melnikov antenna array (from Proposition 2)

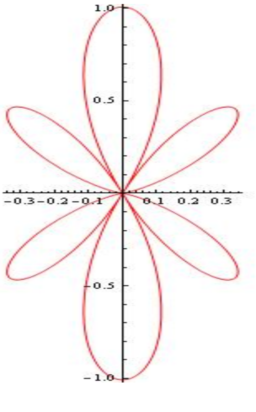

Figure 8. A typical Melnikov antenna array (from Proposition 5)

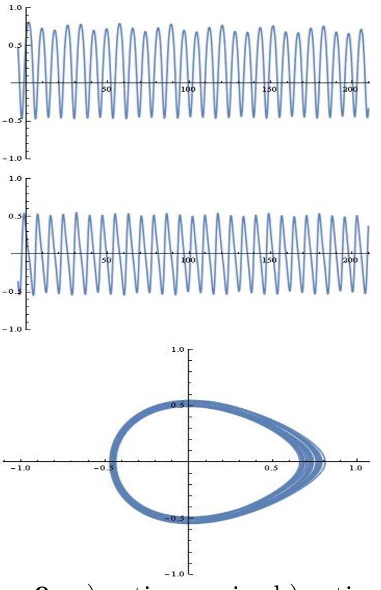

Figure 9. a) x-time series b) y-time series c) phase space (Example 1)

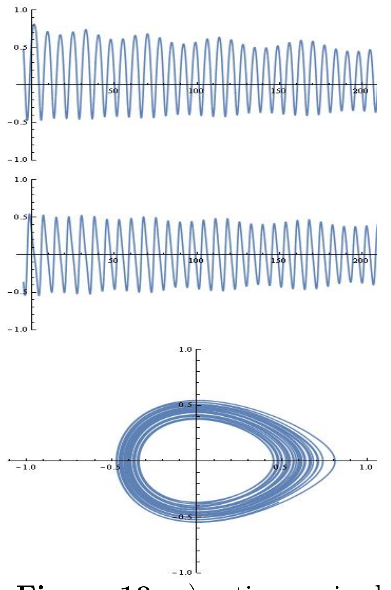

Figure 10. a) x-time series b) y-time series c) phase space (Example 2)

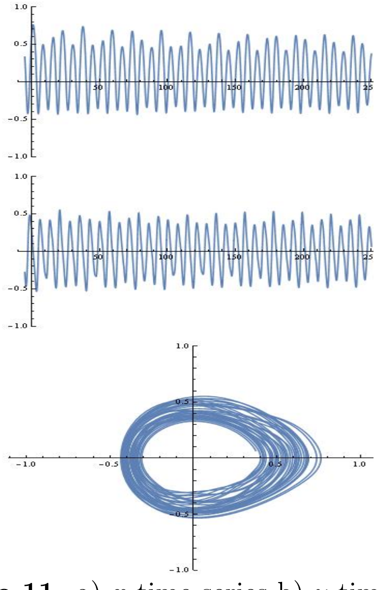

Figure 11. a) x-time series b) y-time series c) phase space (Example 3)

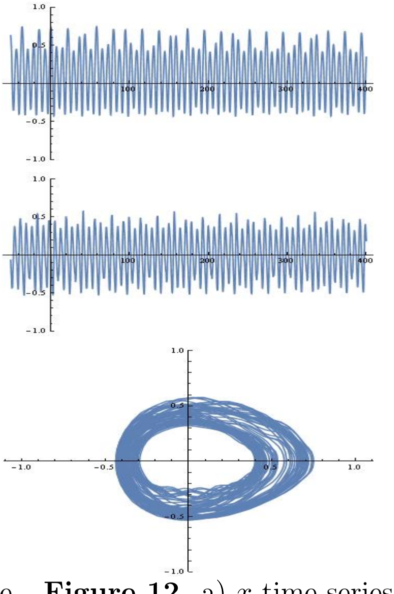

Figure 12. a) x-time series b) y-time series c) phase space (Example 4)

Example 1. For given $p = 2$, $N = 4$, $A = 0.01$, $\epsilon = 0.1$, $\omega = 1.01$, $g_1 = 0.09$, $g_2 = 0.06$, $g_3 = 0.03$, $g_4 = 0.005$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.3$ are depicted on fig. 9.

Example 2. For given $p = 4$, $N = 4$, $A = 0.4$, $\epsilon = 0.1$, $\omega = 1.01$, $g_1 = 0.09$, $g_2 = 0.06$, $g_3 = 0.03$, $g_4 = 0.005$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.3$ are depicted on fig. 10.

Example 3. For given $p = 6$, $N = 5$, $A = 1.2$, $\epsilon = 0.1$, $\omega = 1.2$, $g_1 = 0.49$, $g_2 = 0.46$, $g_3 = 0.13$, $g_4 = 0.1$, $g_5 = 0.7$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.3$ are depicted on fig. 11.

Example 4. For given $p = 8$, $N = 7$, $A = 1.2$, $\epsilon = 0.1$, $\omega = 1.2$, $g_1 = 0.6, g_2 = 0.5, g_3 = 0.2, g_4 = 0.1, g_5 = 0.55, g_6 = 0.7, g_7 = 0.01$ the simulations on the system (4) for $x_0 = 0.6$; $y_0 = 0.3$ are depicted on fig. 12.

Example 5. For given $n = 2$, $N = 2$, $A = 0.01$, $\epsilon = 0.1$, $\omega = 1.01$, $g_1 = 0.9, g_2 = 0.6$ the simulations on the system (3) for $x_0 = 0.7$; $y_0 = 0.3$ are depicted on fig. 13.

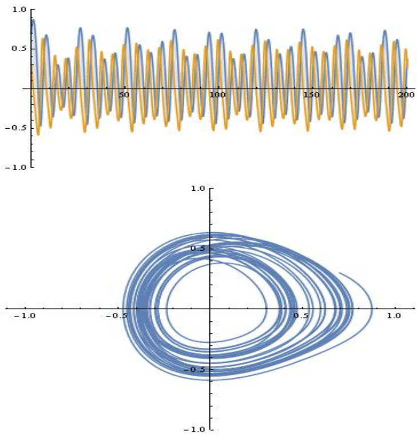

Figure 13. The solutions of the system (3) b) phase space (Example 5)

6. Stochastic control on the parameters

To incorporate a stochastic element in the proposed oscillator, we assume now that the parameters g_j that drives it are the probabilities of some finite discrete distribution. Our choice falls on the following distributions – discrete uniform, binomial, β-binomial, and hypergeometric distributions. We shall use the following well-known exponential presentation of the sin-function

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \tag{14}$$

where i is the imaginary unit. Let the related random variable be denoted by ξ and its characteristic function be $\psi(\cdot)$. Thus we can rewrite the dynamics that drives the oscillator (4) as

$$
\begin{aligned}
\frac{dy}{dt} &= -x + x^2 - \epsilon\left(Ay|y|^{p-1} + \sum_{j=1}^{N} g_j \sin(j\omega t)\right) \\
&= -x + x^2 - \epsilon\left(Ay|y|^{p-1} + \sum_{j=1}^{N} g_j \frac{e^{ij\omega t} - e^{-ij\omega t}}{2i}\right) \\
&= -x + x^2 - \epsilon Ay|y|^{p-1} + \frac{\epsilon}{2i}\mathbb{E}\left[e^{i\xi\omega t}\right] - \frac{\epsilon}{2i}\mathbb{E}\left[e^{-i\xi\omega t}\right] \\
&= -x + x^2 - \epsilon Ay|y|^{p-1} + \frac{\epsilon}{2i}\psi\left(\omega t\right) - \frac{\epsilon}{2i}\psi\left(-\omega t\right).
\end{aligned}
\tag{15}
$$

The above-mentioned distributions lead to the following values for the coefficients g_j with superscript u for uniform, b for binomial, β for β-binomial and h for hypergeometric:

$$
\begin{gathered}
g_j^{\mathrm{u}} = \frac{1}{K}, g_j^{\mathrm{b}} = \binom{k}{j} p^j \left(1-p\right)^{K-j}, g_j^{\mathrm{h}} = \frac{\binom{K}{j}\binom{N-K}{M-j}}{\binom{N}{M}}, \\
g_j^{\beta} = \binom{K}{j} \frac{B\left(j+\alpha, k-j+\beta\right)}{B\left(\alpha,\beta\right)},
\end{gathered}
\tag{16}
$$

where $0 < p < 1$, $\alpha > 0$, $\beta > 0$, $K \leq M \leq \left[\frac{N}{2}\right]$, and M and N are integers. The β-function $B\left(\cdot,\cdot\right)$ is defined via the gamma function as

$$
B\left(x,y\right) = \frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)}. \tag{17}
$$

The characteristic functions of these distributions are

$$
\begin{aligned}
&\psi^{\mathrm{u}}\left(x\right) = \frac{1 - e^{i(K+1)x}}{K\left(1 - e^{ix}\right)}, \psi^{\mathrm{b}}\left(x\right) = \left(1 - p + pe^{ix}\right)^K, \\
&\psi^{\beta}\left(x\right) = {}_2F_1\left(-K, \alpha, \alpha+\beta, 1 - e^{ix}\right), \\
&\psi^{\mathrm{h}}\left(x\right) = \frac{\binom{N-K}{M}{}_2F_1\left(-M, -K, N-M-K+1, e^{ix}\right)}{\binom{N}{M}}.
\end{aligned}
\tag{18}
$$

We express oscillator's dynamics estimating characteristic functions (18) into formula (15).

In addition we can extend the model (4) admitting infinite sum, i.e. $N \to \infty$. Thus the support of the random variable ξ is the set of integer numbers. We suggest using the geometric distribution – it is the unique distribution that exhibits the property memoryless. The probabilities and the characteristic function of this distribution are

$$g_j = (1-p)^{j-1}\, p$$
$$\psi^{\text{geometric}}(x) = \frac{pe^{ix}}{1-(1-p)\, e^{ix}}, \tag{19}$$

where p is again a positive constant less than one.

7. Concluding Remarks

We have investigated in this paper a generalized differential model for escape oscillators in the light of the Melnikov's approach. Also, some dynamic modules implemented in CAS Wolfram Mathematica for investigating the dynamics of the new hypothetical oscillators have been demonstrated. Nonstandard numerical algorithms connected to deriving the roots of the nonlinear equation $M(t_0) = 0$ can be found in (Iliev & Kyurkchiev 2010). We have to note that the calculation of homoclinic and heteroclinic Melnikov integrals as well as the corresponding criterion for the chaos occurrence in the dynamical system are problematic tasks for users of CAS for scientific computing. This necessitates the professional upgrading of the existing specialized modules. Other algorithms (hidden to the user) used in this article:

(1) for arbitrary values of the parameters of the model fixed by the user – the production of the extended oscillator;

(2) algorithm for matching the initial approximations when solving the differential systems, given its interesting specificity and behavior of the solution in confidential time intervals;

(3) algorithms for generating the Melnikov functions of higher kind for analysis of homo/hetero clinic bifurcation in mechanical system.

The offered modules are only part of the more general project for investigating nonlinear models. We are developing a high-scalable, cloud-based software calculator using serverless architecture (Rahneva & Pavlov 2021). The serverless architecture enables automatic scaling of the system during high load. Furthermore, it can be used to parallelize suitable computations for higher efficiency. Where possible, we employ various optimization techniques for high-performance calculations, including multi-processor and multi-threading calculations, and hardware intrinsics (Pavlov 2021; Duffi 2009; Miller 1975). The system is exposing the implemented algorithms using industry-standard application programming interface using HTTP and REST, with data being serialized in JSON and XML formats. The API can be used by reporting and analytics systems like PowerBI and Excel to further investigate the results (Mateev 2022). They will be an integral part of the mentioned above Web-based application for scientific computing. For some details see (Kyurkchiev et al. 2023, 2024, 2024a; Kyurkchiev & Iliev 2022; Golev et al. 2024) . In our application, we also use an algorithm for control and visualization of the antenna factor (with a possibly user-set value of the lateral radiation). It is based on the research in the articles (Kyurkchiev & Andreev 2013) and (Apostolov 2023).

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