Educational Research
Научно-методически статии

COMPUTER DISCOVERED MATHEMATICS: PEDAL CORNER PRODUCTS

[1]Sava Grozdev, Deko Dekov
[1]University of Finance, Business and Entrepreneurship

Abstract. Theorems about pedal corner products obtained by the computer program "Discoverer" are presented in the paper. 2010 Mathematics Subject Classification: Primary 51-04, 68T01, 68T99.

Keywords: pedal corner product, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, Discoverer.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See e.g. (Grozdev & Dekov, 2014 a,b, 2015). In this paper, by using the "Discoverer", we investigate the pedal corner products. The paper contains more than 100 theorems about pedal corner products. We expect that the majority of these theorems are new, discovered by a computer.

We refer the reader to (Kimberling, Glossary) for the definition of a triangle center.

Let *P* and *Q* be finite triangle centers. Let *PaPbPc* be the pedal triangle of *P*. Denote by *Ha* the *Q*-triangle center wrt $\triangle APcPb$, by *Hb* the *Q*-triangle center wrt $\triangle BPaPc$, and by *Hc* is the *Q*-triangle center wrt $\triangle CPbPa$. If the lines *AHa*, *BHb* and *CHc* concur in a point, we say that the *pedal corner product of P and Q exists*, and we call the point of concurrence of the lines the *pedal corner product of P and Q*.

The computer program "Discoverer" has discovered the following theorems:

Theorem 1. *The Pedal Corner Product of a finite triangle center P and the Circumcenter exists, and it is Point P.*

Theorem 2. *The Pedal Corner Product of a finite triangle center P and the Orthocenter exists, and it is the Isogonal Conjugate of Point P.*

Figure 1 illustrates theorem 2. In Fig.1., *P* is an arbitrary point, *PaPbPc* is the pedal triangle of *P*, *Ha* is the orthocenter of $\triangle APcPb$, *Hb* is the orthocenter of $\triangle BPaPc$, and *Hc* is the orthocenter of $\triangle CPbPa$. Then the lines *AHa*, *BHb* and *CHc* concur in point *gH*, the isogonal conjugate of *P*.

In this paper we give a proof of theorem 2 by using barycentric coordinates. Also, we give examples of pedal corner products, discovered by the "Discoverer".

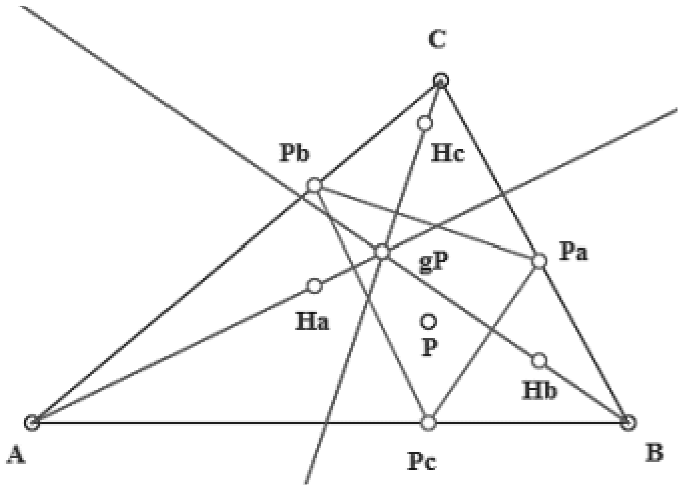

Fig. 1.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to (Grozdev & Nenkov, 2012 a,b), (Paskalev & Tchobanov, 1985).

Given triangle *ABC*, the side lengths are denoted by $a = BC$, $b = CA$ and $c = AB$. The labeling of triangle centers follows (Kimberling). Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, etc.

We use barycentric coordinates. The reference triangle *ABC* has vertices $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

for $\forall k \in \mathbb{R} \setminus \{0\}$: $P = (u,v,w)$ means that $P \simeq (u,v,w) \simeq (ku,kv,kw)$.

A point $P = (u,v,w)$ is *finite*, if $u+v+w \neq 0$. A finite point $P = (u,v,w)$ is *normalized*, if $u+v+w=1$. A finite point could be put in normalized form by $P = \left(\frac{u}{s}, \frac{v}{s}, \frac{w}{s}\right)$, where $s = u+v+w$. The vertices of the pedal triangle of $P = (u,v,w)$ have barycentric coordinates:

$$Pa = (0, -c^2u + a^2u + b^2u + 2a^2v, -b^2u + c^2u + a^2u + 2a^2w),$$

$$Pb = (-c^2v + a^2v + b^2v + 2b^2u, 0, -a^2v + b^2v + c^2v + 2b^2w),$$

$$Pc = (-b^2w + c^2w + a^2w + 2c^2u, -a^2w + b^2w + c^2w + 2c^2v, 0).$$

Given two normalized points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$, then (Paskalev & Tchobanov, 1985, § 15, Proposition 1):

$$|PQ|^2 = -a^2vw - b^2wu - c^2uv, \tag{1}$$

where $u = u_1 - u_2$, $v = v_1 - v_2$ and $w = w_1 - w_2$.

Let *DEF* be a triangle whose vertices have normalized barycentric coordinates wrt $\triangle ABC$ as it follows: $D = (p_1, q_1, r_1)$, $E = (p_2, q_2, r_2)$ and $F = (p_3, q_3, r_3)$. Let *P* be a point with normalized barycentric coordinates $P = (p, q, r)$ wrt $\triangle DEF$. Then the barycentric coordinates of $P = (u, v, w)$ wrt $\triangle ABC$ are as it follows (Paskalev & Tchobanov, 1985, § 30):

$$\begin{aligned} u &= p_1p + p_2q + p_3r \\ v &= q_1p + q_2q + q_3r \\ w &= r_1p + r_2q + r_3r \end{aligned} \tag{2}$$

The equation of a line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{3}$$

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{4}$$

The intersection of two lines $L_1 : p_1x + q_1y + r_1z = 0$ and $L_2 :\ p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1). \tag{5}$$

Given a point $P = (u, v, w)$, the isogonal conjugate of *P* is the point (a^2vw, b^2wu, c^2uv).

3. Proof of Theorem 2

Proof. Given $\triangle ABC$. Let $P = (u, v, w)$ be a finite triangle center of $\triangle ABC$ and let *PaPbPc* be the pedal triangle of *P*. By using (1) we compute the side lengths $a_1 = |PbPc|$, $b_1 = |APb|$ and $c_1 = |APc|$ of $\triangle APcPb$. The barycentric coordinates of the orthocenter $Ha = (uHa, vHa, wHa)$ of $\triangle APcPb$ wrt $\triangle APcPb$ are as it follows:

$$Ha = [a_1^4 + (b_1^2 - c_1^2)^2, b_1^4 + (c_1^2 - a_1^2)^2, c_1^4 + (a_1^2 - b_1^2)^2].$$

Hence

$$uHa = vw(a+b+c)(b+c-a)(c+a-b)(a+b-c),$$
$$vHa = w(a^2 - c^2 - b^2)(a^2 v - b^2 v - c^2 v - 2b^2 w),$$
$$wHa = v(a^2 - c^2 - b^2)(a^2 w - b^2 w - c^2 w - 2c^2 v).$$

By using (2), we find the barycentric coordinates of *Ha* wrt $\triangle ABC$ as it follows:

$$uHa = a^2c^2v + a^2b^2w + b^2c^2v + b^2c^2w + 2b^2c^2u - b^4w - c^4v,$$

$$vHa = b^2w(b^2 + c^2 - a^2),\ wHa = c^2v(b^2 + c^2 - a^2).$$

Similarly, we find the barycentric coordinates of *Hb* and *Hc* wrt $\triangle ABC$. Then, by using (3), we find the barycentric equations of the lines *AHa*, *BHb* and *CHc* as it follows:

$$AHa\text{: } c^2vy - b^2wz = 0,\quad BHb\text{: } c^2ux - a^2wz = 0,\quad CHc\text{: } b^2ux - a^2vy = 0.$$

By using (4), we prove that these lines concur in a point. Then, by using (5), we find the point of intersection of the lines *AHa*, *BHb* and *CHc* as the point of intersection $Q = (uQ, vQ, wQ)$ of the lines *AHa* and *BHb* as it follows:

$$uQ = a^2vw,\ vQ = b^2wu,\ wQ = c^2uv.$$

Point *Q* is the pedal corner product of point *P* and the orthocenter.

We see that point *Q* coincides with the isogonal conjugate of point *P*. This completes the proof.

4. New properties of notable points of the triangle

The computer program “Discoverer” has produced 174 examples of pedal corner products. Of these 89 are points which are available in (Kimberling) and the rest of 85 points are not available in (Kimberling). Clearly, the number of examples could be easily extended by the “Discoverer”.

We may use the enclosed List K (or equivalently, the enclosed tables Table P-X, or Table X-P) in order to add new theorems to the corresponding articles in the encyclopedias.

Below we give an example. Consider the row 28 of Table X-P. We can rewrite the row to the following theorem:

Theorem 3. *The Pedal Corner Product of the Outer Fermat Point and the Kosnita Point is the Outer Napoleon Point.*

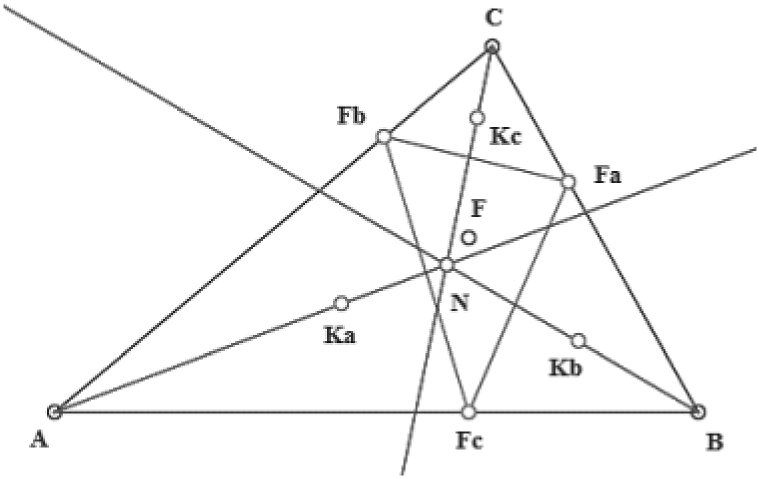

Fig. 2.

Figure 2 illustrates theorem 3. In Fig. 2, *F* is the Outer Fermat point, *FaFbFc* is the pedal triangle of *F*, *Ka* is the Kosnita point of $\triangle AFcFb$, *Kb* is the Kosnita point of $\triangle BFaFc$, and *Kc* is the Kosnita point of $\triangle CFbFa$. Then the lines *AKa*, *BKb* and *CKc* concur in point *N*, the Outer Napoleon point.

5. New notable points of the triangle

We may use the results in the enclosed List D in order to define new remarkable points of the triangle. We may expect that the pedal corner products, available in List D, are new remarkable points, because they are not included in the (Kimberling).

As example, consider row 21 in List D. Recall that the X(69) the Retrocenters is the Symmedian point of the Antimedial triangle. The row rewrites to the following theorem:

Theorem 4. *The Pedal Corner Product of the Bevan Point and the Retrocenter exists.*

Figure 3 illustrates theorem 4. In Fig. 3, *P* is the Bevan point, *PaPbPc* is the pedal triangle of the Bevan point, *Ra* is the Retrocenter of $\triangle APcPb$, *Rb* is the Retrocenter of $\triangle BPaPc$, and *Rc* is the Retrocenter of $\triangle CPbPa$. Then, the lines *ARa*, *BRb* and *CRc* concur in a point.

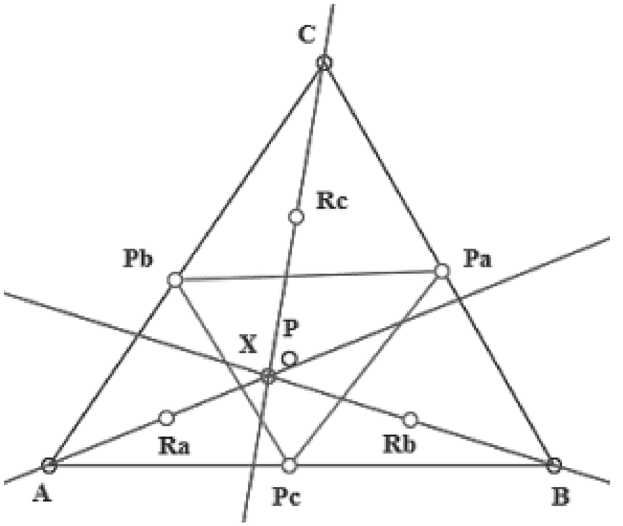

Fig. 3.

We can now define the point “Pedal Corner Product of the Bevan Point and the Retrocenter” as a new remarkable point of the triangle. The barycentric coordinates of the new point are as follows: $f(a,b,c)$, $f(b,c,a)$ and $f(c,a,b)$, where

$$f(a,b,c) = -a(a-b-c)^2(b^2c+bc^2+a^2b-ab^2+a^2c-ac^2-2abc+a^3-b^3-c^3)$$

.

Supplementary material

The enclosed file “2015-pcp.zip” contains the files quoted in this paper. The reader may download it from the web site of the journal..

REFERENCES

Grozdev, S. & Dekov, D. (2014 a). Computer-generated mathematics: Points on the Kiepert hyperbola, *The Mathmatical Gazette*, vol. 98, no. 543, 509 – 511.

Grozdev S. & Dekov D. (2014 b). The Computer Program “Discoverer” and the Encyclopedia of Computer-Generated Mathematics (in Bulgarian), *International Journal of Computer-Generated Mathematics*, vol 9, no. 2, http://www.ddekov.eu/j/index.htm

Grozdev, S. & Dekov, D. (2015). A Computer Improves the Steiner’s Construction of the Malfatti Circles, *Mathematics and Infomatics*, vol. 58, no.1, 40 – 51.

Grozdev, S. & Nenkov, V. (2012a). *Three Remarkable Points on the Medians of a Triangle* (in Bulgarian), Sofia, Archimedes.

Grozdev, S. & Nenkov, V. (2012b). *On the Orthocenter in the Plane and in the Space* (in Bulgarian), Sofia, Archimedes.

Kimberling, C., *Encyclopedia of Triangle Centers*, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

Paskalev, G. & Tchobanov, I. (1985). Remarkable Points in the Triangle (in Bulgarian), Sofa, Narodna Prosveta.

✉ **Prof. Sava Grozdev, DSc**
University of Finance, Business and Entrepreneurship
1, Gusla Street
1618 Sofia, Bulgaria
E-mail: sava.grozdev@gmail.com

✉ **Dr. Deko Dekov, Assoc. Prof.**
81, Zahari Knjazheski Street
6000 Stara Zagora, Bulgaria
E-mail: ddekov@ddekov.eu