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COMPUTER DISCOVERED MATHEMATICS: ORTHOLOGY CENTERS OF THE EULER TRIANGLES

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Abstract. The paper studies the orthologic triangles of the Euler triangles and their orthology centers. The results are discovered by the computer program "Discoverer".

Keywords: Euler triangles, orthologic triangles, orthology center, computer-generated mathematics, Discoverer.

1. Introduction

In the recent years many theorems are discovered by experimenting with dynamical geometry software. See e.g. (Baralić et al., 2014). But the possibilities of this approach are limited. The next step is the use of computers-discoverers. In this paper we present the computer-program "Discoverer", created by the authors. The computer program "Discoverer" is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. The computer program discovers theorems in Euclidean geometry. The "Discoverer" was able to discover an essential improvement of the classical Steiner's solution of the construction of the Malfatti circles (see (Grozdev & Dekov, 2015)), as well as a number of theorems in Euclidean geometry see e.g. (Grozdev & Dekov, 2014a,b).

The orthologic triangles are studied since 1827 when Jacob Steiner discovered some basic facts about them. In the same year August Ferdinand Möbius discovered the barycentric coordinates. In this paper we present some results discovered by the "Discoverer" and related to the orthologic triangles and orthology centers. All theorems and examples, as well as their proofs, are produced by the computer program "Discoverer".

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to (Grozdev & Nenkov, 2012a,b), (Stanilov, 1979), (Yiu, 2013).

Given a triangle *ABC* with side lengths $BC = a$, $CA = b$ and $AB = c$. The barycentric coordinates with respect to ΔABC are used to define points, lines, circles, triangles, etc. The reference triangle *ABC* has vertices $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is:

For all $k \in \mathbb{R} - \{0\}$: $P = (u, v, w)$ means that $P \cong (u, v, w) \cong (ku, kv, kw)$.

We use the Conway triangle notation:

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2),\ S_B = \frac{1}{2}(c^2 + a^2 - b^2),\ S_C = \frac{1}{2}(a^2 + b^2 - c^2),$$

$S = 2\Delta$,

where Δ is the area of the reference triangle.

The equation of the line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \qquad (1)$$

The infinite point of the line *L*: $px + qy + rz = 0$ is the point (f, g, h) = $(q - r, r - p, p - q)$. Denote $F = S_B g - S_C h$, $G = S_C h - S_A f$ and $H = S_A f - S_B g$. Then the line through the point $P = (u, v, w)$ and perpendicular to the line *L*: $px + qy + rz = 0$ has equation:

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \qquad (2)$$

Three lines $p_i x + q_i y + r_i z = 0$, $i = 1, 2, 3$ are concurrent iff

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \qquad (3)$$

The intersection of the two lines $L_1 : p_1 x + q_1 y + r_1 z = 0$ and L_2: $p_2 x + q_2 y + r_2 z = 0$ is the point

$$(q_1 r_2 - q_2 r_1, r_1 p_2 - r_2 p_1, p_1 q_2 - p_2 q_1). \qquad (4)$$

If the first barycentric coordinate of a triangle center $R=(uR,vR,wR)$ is $uR=f(a,b,c,u,v,w)$, then $vR=f(b,c,a,v,w,u)$ and $wR=f(c,a,b,w,u,v)$. Hence, in order to define the triangle center *R* it is enough to define the first barycentric coordinate *uR*.

Given ΔABC and an arbitrary point *P*, the Euler triangle of *P* is the triangle whose vertices E_1, E_2, and E_3 are the midpoints of the segments *AP*, *BP* and *CP*, respectively. The Euler triangle of the orthocenter is the ordinary Euler Triangle (See (Weisstein[3)], Euler Triangle). The vertices of the Euler triangle $E_1E_2E_3$ of point $P=(u,v,w)$ have barycentric coordinates, See (Grozdev & Dekov, 2014b):

$E_1=(2u+v+w,v,w)$, $E_2=(u,u+2v+w,w)$ and $E_3=(u,v,u+v+2w)$.

The $\Delta A_1B_1C_1$ is orthologic wrt $\Delta A_2B_2C_2$ if the perpendiculars from the vertices A_1, B_1, C_1 to the sides B_2C_2, C_2A_2, A_2B_2 are concurrent. If this is the case, the point of intersection of the perpendiculars is known to be the orthology center of $\Delta A_1B_1C_1$ wrt $\Delta A_2B_2C_2$. In 1827 Steiner showed that if $\Delta A_1B_1C_1$ is othologic wrt $\Delta A_2B_2C_2$, then the converse is true, that is, $\Delta A_2B_2C_2$ is orthologic wrt $\Delta A_1B_1C_1$. For examples of orthologic triangles see e.g. (Danneels & Dergiades, 2004), (Gibert[1)], Table 7).

In this paper we denote by *E*(*T*, *P*) the orthology center of the Euler Triangle of point *P* wrt triangle *T*, and by *C*(*T*, *P*) the orthology center of triangle *T* wrt the Euler Triangle of point *P*.

The labeling of triangle centers follows ETC (Kimberling[2)]). Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, etc. For the definition of various triangles and the barycentric coordinates of their vertices we refer the reader to (Weisstein[3]). Note that the computer program "Discoverer" uses the term "Inner (Outer) Yff triangle" instead of "First (Second) Yff circles triangle".

3. Orthology Centers of the Euler Triangles wrt Triangle *ABC*

Theorem 3.1. *For any point* $P=(u,v,w)$*, the Euler triangle of P is orthologic wrt* ΔABC. *The first barycentric coordinate of the orthology center* $R=(uR,vR,wR)$ *is*

$$uR=va^4+wa^4-vb^4-wb^4-vc^4-wc^4-2ub^4-2uc^4$$
$$+4uc^2b^2+2ua^2c^2+2ua^2b^2+2vb^2c^2+2wb^2c^2.$$

Proof. The equation of the line *BC* is $x=0$. By using (1) we easily obtain this equation. The infinite point of the line *BC* is $(0,-1,1)$, so that the infinite point of any line, perpendicular to *BC*, is $(2a^2,c^2-a^2-b^2,b^2-c^2-a^2)$. Next we calculate the equation of the line L_1 through the point $E_1=(2u+v+w,v,w)$ and perpendicular to the line *BA*. By using (2), we obtain L_1: $p_1x+q_1y+r_1z=0$, where

$$p_1 = wa^2 - va^2 + vb^2 + wb^2 - vc^2 - wc^2$$
$$q_1 = 3wa^2 - 2ub^2 + 2uc^2 + 2ua^2 - vb^2 +'vc^2 + va^2 - wb^2 + wc^2$$
$$r_1 = -3va^2 - 2ua^2 + 2uc^2 - 2ub^2 - vb^2 + vc^2 - wa^2 - wb^2 + wc^2.$$

The equations of the line L_2 through the point E_2 and perpendicular to the line *CA*, is $L_2 : p_2x + q_2y + r_2z = 0$, where

$$p \ = -3wb^2 + 2va^2 - 2vb^2 - 2vc^2 + ua^2 + wa^2 - ub^2 - uc^2 - wc^2,$$
$$q_2 = -wa^2 - ua^2 + ub^2 - wb^2 + uc^2 + wc^2,$$
$$r_2 = 3ub^2 - 2vc^2 + 2vb^2 + 2va^2 + ua^2 - uc^2 - wc^2 + wa^2 + wb^2$$

The equations of the line L_3 through the point E_3 and perpendicular to the line *AB*, is $L_3 : p_3x + q_3y + r_3z = 0$, where

$$p_3 = 3vc^2 - 2wa^2 + 2wb^2 + 2wc^2 - ua^2 - va^2 + ub^2 + vb^2 + uc^2,$$
$$q_3 = -3uc^2 - 2wa^2 + 2wb^2 - 2wc^2 + ub^2 - ua^2 + vb^2 - vc^2 - va^2,$$
$$r_3 = -uc^2 + va^2 + ua^2 - ub^2 + vc^2 - vb^2$$

We use (3) in order to prove that the three lines L_1, L_2 and L_3 concur in a point. Finally, we use (4) to find the barycentric coordinates of the point of concurrence *R*, that is, the orthology center, as the point of concurrence of the lines L_1 and L_2. The barycentric coordinates of the point *R* are given in the statement of the theorem.

Theorem 3.1 defines a transform *E*(Triangle *ABC*, *P*) = *R* of the plane of the Triangle *ABC*. If *T* is any triangle homothetic with the triangle *ABC*, then for any point *P*, *E*(*T*, *P*) = *E*(Triangle *ABC*, *P*). We remind that the following triangles are homothetic with the triangle *ABC*: Medial triangle, Antimedial triangle, Johnson triangle, Inner Yff triangle, Outer Yff triangle, Half-Median triangle, etc. Hence, *E*(Triangle *ABC*, *P*) = *E*(Medial triangle, *P*), etc. Note that if we want directly to calculate e.g. the *E*(Inner Yff triangle, *P*), we have to perform much complicated calculation, but finally we will obtain *E*(Inner Yff triangle, *P*) = *E*(Triangle *ABC*, *P*).

In Table 1 below *T* is the ΔABC or any triangle homothetic with ΔABC.

Table 1

	P	**E(T, P)**
1	X(1) Incenter	X(946)
2	X(2) Centroid	X(381) Center of the Orthocentroidal Circle
3	X(3) Circumcenter	X(5) Nine-Point Center
4	X(4) Orthocenter	X(4) Orthocenter
5	X(5) Nine-Point Center	X(546)
6	X(6) Symmedian Point	X(5480)
7	X(8) Nagel Point	X(355) Center of the Fuhrmann Circle

8	X(13) Outer Fermat Point	X(5478)
9	X(14) Inner Fermat Point	X(5479)
10	X(20) de Longchamps Point	X(3) Circumcenter
11	X(24) Perspector of the Kosnita Triangle and the Orthic Triangle	X(235)
12	X(25) Product of the Orthocenter and the Symmedian Point	X(1596)
13	X(40) Bevan Point	X(10) Spieker Center
14	X(52) Orthocenter of the Orthic Triangle	X(5446)
15	X(54) Kosnita Point	X(3574)
16	X(69) Retrocenter	X(1352)
17	X(74) Ceva Product of the First Isodynamic Point and the Second Isodynamic Point	X(125)
18	X(98) Tarry Point	X(115) Kiepert Center
19	X(99) Steiner Point	X(114)
20	X(100) Anticompliment of the Feuerbach Point	X(119)
21	X(110) Euler Reflection Point	X(113)
22	X(111) Parry Point	X(5512)
23	X(381) Center of the Orthocentroidal Circle	X(3845)
24	X(1141) Gibert Point	X(137)

Remark 3.2. For any point *P*:
(1) *C*(Triangle *ABC*, *P*) = X(4) Orthocenter;
(2) *C*(Medial triangle, *P*) = X(3) Circumcenter;
(3) *C*(Antimedial triangle, *P*) = X(20) de Longchamps Point;
(4) *C*(Johnson triangle, *P*) = X(3) Circumcenter;
(5) *C*(Inner Yff triangle, P) = X(1478) Center of the Inner Johnson-Yff Circle;
(6) *C*(Outer Yff triangle, *P*) = X(1479) Center of the Outer Johnson-Yff Circle.

4. Orthology Centers of the Euler Triangles wrt the Excentral Triangle

Theorem 4.1. *For any point* $P=(u,v,w)$*, the Euler triangle of P is orthologic wrt the Excentral triangle. The first barycentric coordinate of the orthology center* $R=(uR,vR,wR)$ *is* $uR=(2a+b+c)u+av+aw$.

Proof. We use the algorithm of the proof of theorem 3.1. The vertices of the Excentral triangle $J_aJ_bL_c$ have barycentric coordinates $J_a=(-a,b,c)$, $J_b=(a,-b,c)$ and $J_c=(a,b,-c)$. The equation of the line J_bJ_c is $cy+bz=0$, the infinite point of the line J_bJ_c is $(c-b,b,-c)$, so that the infinite point of any line, perpendicular to J_bJ_c is $(b+c,-b,-c)$. The equation of the line L_1 through the point $E_1 = (2u \quad v \quad w,v,w)$ and perpendicular to the line J_bJ_c is

$$L_1 : (-bw + cv)x - (bw + cv + 2cu + 2cw)y + (bw + cv + 2bu + 2bv)z = 0 .$$

Similarly, we calculate the equations of the line L_2 through the point E_2 and perpendicular to the line J_cJ_a, and the line L_3 through the point E_3 and perpendicular to the line J_aJ_b:

$$L_2 : -(uc + wa + 2cv + 2cw)x + (uc - wa)y + (uc + wa + 2au + 2av)z = 0 ,$$

$$L_3 : (av + ub + 2bv + 2bw)x - (av + ub + 2au + 2wa)y + (av - ub)z = 0 .$$

By using (3) we prove that these three lines concur in a point. Finally, we use (4) to find the barycentric coordinates of the point of concurrence *R* as the point of concurrence of the lines L_1 and L_2. The barycentric coordinates are given in the statement of the theorem.

In Table 2 below, *T* is the Excentral triangle or any triangle homothetic with it. Recall that the following triangles are homothetic with the Excentral triangle: Intouch triangle, Hexyl triangle, Yff Central triangle, etc.

Table 2

	P	E(T, P)
1	X(1) Incenter	X(1) Incenter
2	X(2) Centroid	X(551)
3	X(3) Circumcenter	X(1385)
4	X(4) Orthocenter	X(946)
5	X(6) Symmedian Point	X(1386)
6	X(7) Gergonne Point	X(5542)
7	X(8) Nagel Point	X(10) Spieker Center
8	X(9) Mittenpunkt	X(1001)
9	X(10) Spieker Center	X(1125)
10	X(11) Feuerbach Point	X(1387)
11	X(20) de Longchamps Point	X(4297)
12	X(35) Perspector of the Intangents Triangle and the Kosnita Triangle	X(2646)
13	X(36) Inverse of the Incenter in the Circumcircle	X(1319) Bevan-Schroder Point
14	X(40) Bevan Point	X(3) Circumcenter
15	X(43) Prespector of the Excentral Triangle and the Symmedial Triangle	X(995)
16	X(44) Harmonic Conjugate of the Grinberg Point with respect to the Mittenpunkt and the Symmedian Point	X(3246)

17	X(46) Prespector of the Excentral Triangle and the Orthic Triangle	X(56) External Center of Similitude of Circumcircle and Incircle
18	X(57) Isogonal Conjugate of the Mittenpunkt	X(999)
19	X(63) Isogonal Conjugate of the Clawson Point	X(993)
20	X(65) Orthocenter of the Intouch Triangle	X(942) Nine-Point Center of the Intouch Triangle
21	X(72) Quotient of the Grinberg Point and the Orthocenter	X(960)
22	X(79) Prespector of Triangle ABC and Triangle of Reflections of the Incenter in the Sidelines of Triangle ABC	X(3649)
23	X(80) Reflection of the Incenter in the Feuerbach Point	X(11) Feuerbach Point
24	X(100) Anticompliment of the Feuerbach Point	X(214)
25	X(190) Yff Parabolic Point	X(4432)
26	X(192) Equal Parallelians Point	X(3993)
27	X(354) Weill Point	X(5049)
28	X(355) Center of the Fuhrmann Circle	X(5) Nine-Point Center
29	X(484) Evans Perspector	X(36) Inverse of the Incenter in the Circumcircle
30	X(1019) Center of the Evans Circle	X(4367)
31	X(1155) Schroder Point	X(5126)
32	X(1478) Center of the Inner Johnson-Yff Circle	X(226)

Remark 4.2. For any point *P*:

(1) *C*(Excentral triangle, *P*) = X(40) Bevan Point;

(2) *C*(Intouch triangle, *P*) = X(1) Incenter;

(3) *C*(Hexyl triangle, *P*) = X(1) Incenter;

(4) *C*(Yff Central Triangle, *P*) = Circumcenter of the Yff Central Triangle. This point is not available in (Kimberling[2)]).

5. Orthology Centers of the Euler Triangles wrt the Tangential Triangle

Theorem 5.1. *For any point* $P=(u,v,w)$*, the Euler triangle of P is orthologic wrt the Tangential triangle. The first barycentric coordinate of the orthology center* $R=(uR,vR,wR)$ *is*

$uR = a^4(2u+v+w)+(b^2-c^2)^2u-(a^2b^2+c^2a^2)(3u+v+w)$.

To prove theorem 5.1 we use the same algorithm as in the proofs of the previous theorems. Recall that the vertices of the Tangential triangle $T_aT_bT_c$ are $T_a = (-a^2, b^2, c^2)$, $T_b = (a^2, -b^2, c^2)$ and $T_c = (a^2, b^2, -c^2)$.

In Table 3 below, *T* is the Tangential triangle or any triangle homothetic with it. Recall that the following triangles are homothetic with the Tangential triangle: Orthic triangle, Intangents triangle, Extangents triangle, etc.

Table 3

	P	E(T, P)
1	X(1) Incenter	X(1385)
2	X(2) Centroid	X(549)
3	X(3) Circumcenter	X(3) Circumcenter
4	X(4) Orthocenter	X(5) Nine-Point Center
5	X(5) Nine-Point Center	X(140) Nine-Point Center of the Medial Triangle
6	X(6) Symmedian Point	X(182) Center of the Brocard Circle
7	X(20) de Longchamps Point	X(550)
8	X(21) Schiffler Point	X(5428)
9	X(26) Circumcenter of the Tangential Triangle	X(1658)
10	X(40) Bevan Point	X(3579)
11	X(52) Orthocenter of the Orthic Triangle	X(389) Center of the Taylor Circle
12	X(64) Isogonal Conjugate of the de Longchamps Point	X(3357)
13	X(110) Euler Reflection Point	X(1511)
14	X(182) Center of the Brocard Circle	X(5092)
15	X(355) Center of the Fuhrmann Circle	X(10) Spieker Center
16	X(381) Center of the Orthocentroidal Circle	X(2) Centroid
17	X(399) Parry Reflection Point	X(110) Euler Reflection Point

Remark 5.2. For any point *P*:

(1) *C*(Tangential triangle, *P*) = X(3) Circumcenter;

(2) *C*(Orthic triangle, *P*) = X(4) Orthocenter;

(3) C(Intangents triangle, P) = X(1) Incenter;
(4) C(Extangents triangle, P) = X(40) Bevan Point.

6. Orthology Centers of the Euler Triangles wrt the Malfatti Squares Triangle

Theorem 6.1. *For any point* $P=(u,v,w)$, *the Euler triangle of P is orthologic wrt the Malfatti squares triangle. The first barycentric coordinate of the orthology center* $R=(uR,vR,wR)$ *is*

$$uR = 2(a+b+c)^2(a-b-c)^2(a-b+c)^2(a+b-c)^2 \overline{(-5a^4-5b^4-5c^4+26a^2b^2+26b^2c^2+26c^2a^2)(4u+v+w)} +12(a^2+b^2+c^2)\sqrt{-(a+b+c)(a-b-c)(a-b+c)(a+b-c)}.$$

To prove theorem 6.1 we use the same algorithm as in the proofs of the previous theorems. Recall that the barycentric coordinates of the vertices of the Malfatti squares triangle $M_aM_bM_c$ are $M_a=(S,S+S_A+2S_C,S+S_A+2S_B)$, $M_b=(S+S_B+2S_C,S,S+S_B+2S_A)$ and $M_c=(S+S_C+2S_B,S+S_C+2S_A,S)$.

In Table 4 below, T is the Malfatti squares triangle or any triangle homothetic with it.

Table 4

	P	E(T, P)
1	X(1) Incenter	X(551)
2	X(2) Centroid	X(2) Centroid
3	X(3) Circumcenter	X(549)
4	X(4) Orthocenter	X(381) Center of the Orthocentroidal Circle
5	X(5) Nine-Point Center	X(547)
6	X(6) Symmedian Point	X(597)
7	X(8) Nagel Point	X(3679)
8	X(10) Spieker Center	X(3828)
9	X(13) Outer Fermat Point	X(5459)
10	X(14) Inner Fermat Point	X(5460)
11	X(20) de Longchamps Point	X(376)
12	X(37) Grinberg Point	X(4755)
13	X(69) Retrocenter	X(599)
14	X(75) Moses Point	X(4688)
15	X(99) Steiner Point	X(2482)

16	X(115) Kiepert Center	X(5461)
17	X(190) Yff Parabolic Point	X(4370)
18	X(192) Equal Parallelians Point	X(4664)
19	X(354) Weill Point	X(3742)
20	X(381) Center of the Orthocentroidal Circle	X(5) Nine-Point Center

Remark 6.2. For any point *P*, *C*(Malfatti squares triangle, *P*) = X(3068) Malfatti-Moses Point.

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NOTES

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