Educational Technology
Образователни технологии

COMPUTER-DISCOVERED MATHEMATICS: LALESCO PRODUCTS

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Abstract. The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. In this paper we use "Discoverer" in order to investigate the Lalesco products in the geometry of the triangle.

Keywords. geometry of the triangle, computer-discovered mathematics, Discoverer.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See (Grozdev & Dekov, 2014, 2015). In this paper we give an example of the use of the "Discoverer". We investigate the Lalesco products and we present results obtained by the "Discoverer".

In this paper the labeling of triangle centers follows ETC (Kimberling). Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, etc. Given ΔABC, we denote by a, b and c the side lengths of ΔABC, $a = BC$, $b = CA$ and $c = AB$.

In 2011 Randy Hutson announced the following theorem (See (Kimberling), articles X(99), X(100), X(101), X(1292), X(1293) and X(1296)):

Theorem 1. Let $I = \{(1,2),(1,3),(1,6),(1,7),(2,6),(3,6),(7,8)\}$. Let $(i, j) \in I$ and let L_1 be the reflection of the line X(i)X(j) in line BC, and define L_2 and L_3 cyclically. Let $D = L_2 \cap L_3$, $E = L_3 \cap L_1$, and $F = L_1 \cap L_2$. The lines AD, BE and CF concur in a point denoted by LP(X(i),X(j)). The points of concurrence are as follows: LP(X(1),X(2)) = X(1293), LP(X(1), X(3)) = X(100), LP(X(1), X(6)) = X(1292), LP(X(1),X(7)) = X(101), LP(X(2),X(6)) = X(1296), LP(X(3),X(6)) = X(99) and LP(X(7),X(8)) = X(1292).

Theorem 2. Given ΔABC. Let P and U be points different from the vertices of ΔABC. Let L_1 be the reflection of the line PU in line BC, and define L_2 and L_3 cyclically. Let $D = L_2 \cap L_3$, $E = L_3 \cap L_1$, and $F = L_1 \cap L_2$. The lines AD, BE and CF concur in a point, which lies on the circumcircle of ΔABC.

The construction in theorem 2 is published by T. Lalesco in 1937 and reprinted in 1987. See (Lalesco, 1987).We call *the Lalesco product of P and U* the point of concurrence, defined in theorem 2, and we denote by $LP(P,U)$ the Lalesco product of P and U. Note that the Lalesco product is commutative. Also, if point V lies on the line PU, then $LP(P,V) = LP\,(V,U) = LP(P,U)$.

Figure 1 illustrates the Lalesco product of points P and U.

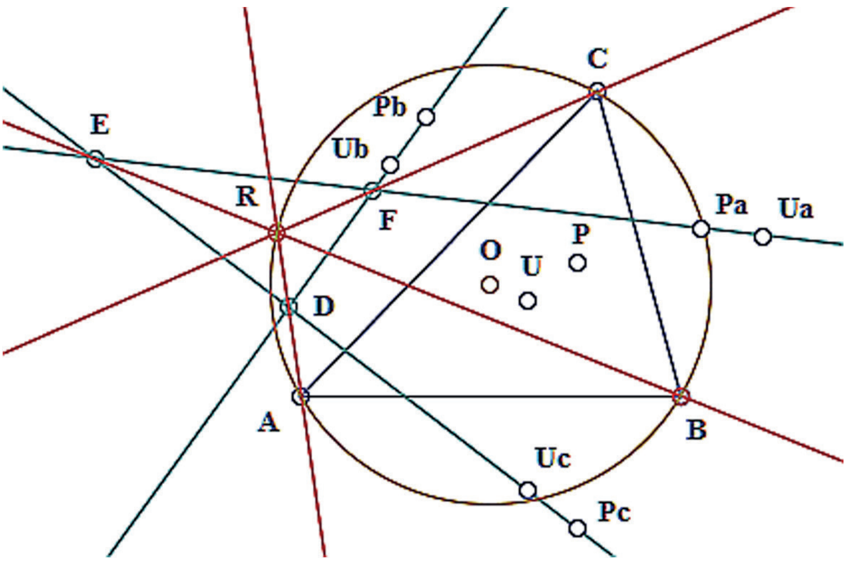

Figure 1. Point R is the Lalesco product of points P and U.

Theorem 3 below is discovered by the computrer program “Discoverer”.

Theorem 3. Given ΔABC. Let P and U be points different from the vertices of ΔABC and having barycentric coordinates $P = (p, q, r)$ and $U = (u, v, w)$. The Lalesco product $LP\,(P, U)$ has barycentric coordinates

$(f\,(a, b, c, p, q, r, u, v, w), (f\,(b, c, a, q, r, p, v, w, u), (f\,(c, a, b, r, p, q, w, u, v))$

where

$$f(a,b,c,p,q,r,u,v,w) = a^2(qc^2w - b^2qw + qub^2 + quc^2 - a^2qu$$
$$-a^2qw - 2pb^2w + b^2vr - vpc^2 + a^2vp + a^2vr + 2ub^2r - vc^2r - vpb^2)$$
$$(qc^2w + a^2qw - b^2qw + 2quc^2 - pb^2w - pc^2w + a^2pw$$
$$-a^2vr + b^2vr + uc^2r - a^2ur - 2vpc^2 + ub^2r - vc^2r)$$

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. For the barycentric coordinates we refer the reader to (Grozdev & Nenkov, 2012a,b).

Given ΔABC. The barycentric coordinates with respect to ΔABC are used to define points, lines, circles, triangles, etc. The reference triangle ABC has vertices $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$. A point is an element of the projective space $P_3(\mathbb{R})$, defined up to a proportionality factor, that is, for $\forall k \in \mathbb{R} \setminus \{0\}$: $P = (u, v, w)$ means that $P \simeq (u, v, w) \simeq (ku, kv, kw)$.

The reflections P_a, P_b and P_c of a point $P = (p, q, r)$ in the sidelines BC, CA, AB, respectively, are the points (See (Hatzipolakis & Yiu, 2009)):.

$$P_a = (-a^2 p, (a^2 + b^2 - c^2)p + a^2 q, (c^2 + a^2 - b^2)p + a^2 r),$$

$$P_b = ((a^2 + b^2 - c^2)q + b^2 p, -b^2 q, (b^2 + c^2 - a^2)q + b^2 r),$$

$$P_c = ((c^2 + a^2 - b^2)r + c^2 p, (b^2 + c^2 - a^2)r + c^2 q, -c^2 r).$$

The equation of the line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0 \cdot \qquad (1)$$

The intersection of two lines $L_1 : p_1 x + q_1 y + r_1 z = 0$ and $L_2 : p_2 x + q_2 y + r_2 z = 0$ is the point

$$(q_1 r_2 - q_2 r_1, r_1 p_2 - r_2 p_1, p_1 q_2 - p_2 q_1). \qquad (2)$$

Three lines $p_i x + q_i y + r_i z = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \quad (3)$$

3. Proof of theorem 3.

Proof. Let $P = (p, q, r)$ and $U = (u, v, w)$ be points distinct from the vertices of ΔABC. Denote by P_a, P_b and P_c the reflections of point P in the sidelines BC, CA and AB, respectively. Similarly, define points U_a, U_b and U_c. By using (1) we obtain that the equation of the line L_1 through points P_a and U_a is $p_1 x + q_1 y + r_1 z = 0$, where

$$p_1 = a^2 ur + ub^2 r - uc^2 r + vpc^2 + a^2 vp - vpb^2 + a^2 vr$$

$$-a^2pw - pb^2w + pc^2w - quc^2 - a^2qu + qub^2 - a^2qw,$$

$$q_1 = a^2(ru - pw),\ r_1 = a^2(-uq + vr).$$

Similarly we obtain $L_2 : p_2x + q_2y + r_2z = 0$ where

$$p_2 = b^2(-vr + qw),$$

$$q_2 = a^2qw + b^2qw - qc^2w + vpb^2 + vpc^2 - a^2vp + pb^2w$$

$$-a^2vr - b^2vr + vc^2r - qub^2 - quc^2 + a^2qu - ub^2r,$$

$$r_2 = b^2(-uq + vp),$$

and $L_3 : p_3x + q_3y + r_3z = 0$, where

$$p_3 = c^2(-vr + qw),\ q_3 = c^2(ru - pw),$$

$$r_3 = qc^2w + a^2qw - b^2qw + ub^2r + uc^2r - a^2ur + quc^2$$

$$-vc^2r - a^2vr + b^2vr - pb^2w - pc^2w + a^2pw - vpc^2.$$

By using (2), we find the barycentric coordinates of points $D = L_B \cap L_C$, $E = L_C \cap L_A$, and $F = L_A \cap L_B$. Next, we obtain that the equations of the line AD is $p_4x + q_4y + r_4z = 0$, where

$$p_4 = 0$$

$$q_4 = c^2(-a^2qw + qc^2w - b^2qw - a^2qu + quc^2 + qub^2 - 2pb^2w$$

$$-vpb^2 - vc^2r + 2ub^2r + a^2vr + a^2vp - vpc^2 + b^2vr),$$

$$r_4 = b^2(qc^2w + a^2qw - b^2qw + 2quc^2 - pb^2w - pc^2w + a^2pw$$

$$-a^2ur + b^2vr - a^2vr + uc^2r - vc^2r - 2vpc^2 + ub^2r).$$

Similarly, we obtain the equations of the lines BE and CF. By using (3), we prove that the lines AD, BE and CF concur in a point. Finally, we find the barycentric coordinates of the point of concurrence $LP\ (P,U)$ as the point of concurrence of the lines AD and BE. The barycentric coordinates of the point $LP\ (P,U)$ are given in the statement of theorem 3.

4. Examples of Lalesco products

The computer program "Discoverer" easily produces examples of Lalesco products. The enclosed HTML-files, containing examples of Lalesco products, are produced by the "Discoverer".

Let **Ω** be the set of all Lalesco products $LP\ (P,U)$ where P is any of the points listed in the enclosed "List 1", and U is any of the first ten points, listed in the same list. The enclosed "List K" contains the Lalesco products $LP(P,U) \in \Omega$ which are available in Kimberling's ETC, and the rest of the points of Ω is listed in the "List D". Also, the "Discoverer" rewrites the List K to two tables. In our investigation "List K" and "List D" contain 387 and 1268 points, respectively.

Any item of "List K" is a theorem. For example, no 62 of the List K is the following

Theorem 4. The Lalesco Product of X(2) Centroid and X(3) Circumcenter is the Euler reflection point X(110).

We can easily prove this theorem by using theorem 3. In the barycentric coordinates of the Lalesco product of *P* and *U*, given in theorem 3 we have to substitute the barycentric coordinates of the Centroid for the barycentric coordinates of *P*, and similarly the barycentric coordinates of the Circumcenter for these of *U*. Recall that the barycentric coordinates of the Centroid G and the Circumcenter O are as follow: G = (1,1,1) and $O=(a^2(b^2+c^2-a^2),b^2(c^2+a^2-b^2),c^2(a^2+b^2-c^2))$. After some simplification of the coordinates, we obtain that the fist coordinate of *LP*(X(2),X(3)) is. $a^2/(b^2-c^2)$. This is the first barycentric coordinate of the Euler reflection point, the point X(110) in (Kimberling).

5. New remarkable points

The enclosed "List D" contains Lalesco products of remarkable points of the triangle which are not available in the ETC. We may expect that these points are new remarkable points of the triangle. By using theorem 3, we can easily find the barycentric coordinates of these new points. For example, let us consider no.126 in the "List D":

Theorem 5. The Lalesco Product of X(2) Centroid and X(37) Grinberg Point is not listed in the ETC.

By using theorem 3, we easily find the barycentric coordinates of this new notable point. Recall that the first barycentric coordinate of the Grinberg Point is $a(b+c)$. The first barycentric coordinate uR of $R = LP$(X(2),X(37)) is as follows:

$$uR = a^2(a-b)(a-c)(2a^2b-a^2c-2abc+2b^2a-b^2c-3c^3)$$
$$(-2a^2c+a^2b+2abc-2ac^2+3b^3+c^2b).$$

Note that in the "List D" there are points which coincide. In the list "List D" are listed 1268 points, but the list contains only 1109 new notable points of a triangle. These new points are listed in the enclosed "List 2".

6. Supplementary material

The enclosed file "Lalesco.zip" contains the files quoted in this paper. The reader may download it from the web page of the journal.

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КОМПЮТЪРНО ОТКРИТАТА МАТЕМАТИКА: ПРОИЗВЕДЕНИЯ НА ЛАЛЕСКО

Резюме. Компютърната програма „Откривател", създадена от авторите, е първата компютърна програма, която лесно може да открива нови теореми в математиката, и може би първата компютърна програма, която лесно може да открива ново знание в науката. В тази статия ние използваме „Откривател", за да изучим произведенията на Lalesco в геометрията на триъгълника.

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