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COMPUTER DISCOVERED MATHEMATICS: ANTIPEDAL CORNER PRODUCTS

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Abstract. By using the computer program "Discoverer", we propose theorems about antipedal corner products.
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1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See (Grozdev & Dekov, 2014, 2015). In this paper, by using the "Discoverer", we investigate the antipedal corner products. The paper contains more than 30 theorems about antipedal corner products. We expect that the majority of these theorems are new theorems, discovered by a computer.

The labeling of triangle centers follows (Kimberling). Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, etc. We refer the reader to (Kimberling, Glossary) for the definition of a triangle center.

Given triangle *ABC*. Let *P* and *Q* be finite triangle centers of $\triangle ABC$ and let $\triangle PaPbPc$ be the antipedal triangle of *P*. We say that $\triangle PaCB$, $\triangle CPbA$ and $\triangle BAPc$ are the *antipedal corner triangles of P*. Denote by *Ha* the *Q*-triangle center of $\triangle PaCB$, by *Hb* the *Q*-triangle center of $\triangle CPbA$, and by *Hc* the *Q*-triangle center of $\triangle BAPc$. If the lines *AHa*, *BHb* and *CHc* concur in a point, we say that the *antipedal corner product of P and Q exists*, and we call the point of concurrence of the lines the *antipedal corner product of P and Q*.

The computer program "Discoverer" has discovered the following theorems:

Theorem 1. *The Antipedal Corner Product of a finite triangle center P and the Orthocenter is the Complement of point P.*

Figure 1 illustrates theorem 1. In Fig.1, *PaPbPc* is the antipedal triangle of point *P*, *Ha* is the Orthocenter of $\triangle PaCB$, *Hb* is the Orthocenter of $\triangle CPbA$, and *Hc* is the Orthocenter of $\triangle BAPc$. Then the lines *AHa*, *BHb* and *CHc* concur in point *cP*, the complement of point *P*.

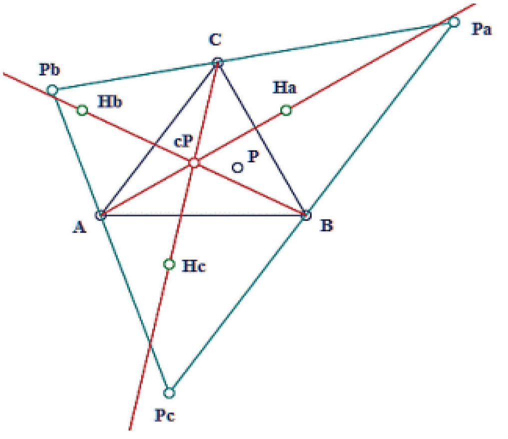

Fig. 1.

Theorem 2. *The Antipedal Corner Product of the Orthocenter and a finite triangle center P is the Complement of point P.*

Figure 2 illustrates theorem 2. In Fig.2, *MaMbMc* is the antipedal triangle of the Orthocenter, that is, the Antimedial triangle, *Pa* is the *P*-triangle center of $\triangle MaCB$, *Pb* is the *P*-triangle center of $\triangle CMbA$, and *Pc* is the *P*-triangle center of $\triangle BAMc$. Then the lines *APa*, *BPb* and *CPc* concur in point *cP*, the complement of point *P*.

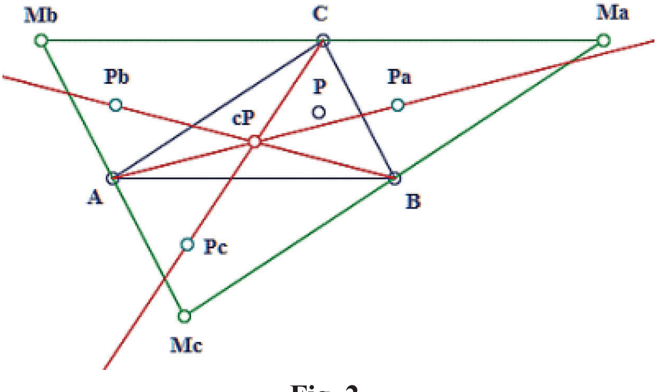

Fig. 2.

Form the proofs of theorems 1 and 2, given below, it follows:

Theorem 3. *The Triangle of the Orthocenters of the Antipedal Corner Triangles of a finite triangle center P is the Triangle of P-triangle centers of the Antipedal Corner Triangles of the Orthocenter.*

Figure 3 illustrates theorem 3 for the special case when *P* is the Circumcenter. In Fig.3, *O* is the Circumcenter, *MaMbMc* is the Antipedal triangle of the Orthocenter, that is, the Antimedial triangle, *PaPbPc* is the Antipedal triangle of the Circumcenter, *Ha* is the Orthocenter of $\triangle PaCB$ and the Circumcenter of $\triangle MaCB$, *Hb* is the Orthocenter of $\triangle CPbA$ and the Circumcenter of the $\triangle CMbA$, and *Hc* is the Orthocenter of $\triangle BAPc$ and the Circumcenter of the $\triangle BAMc$.

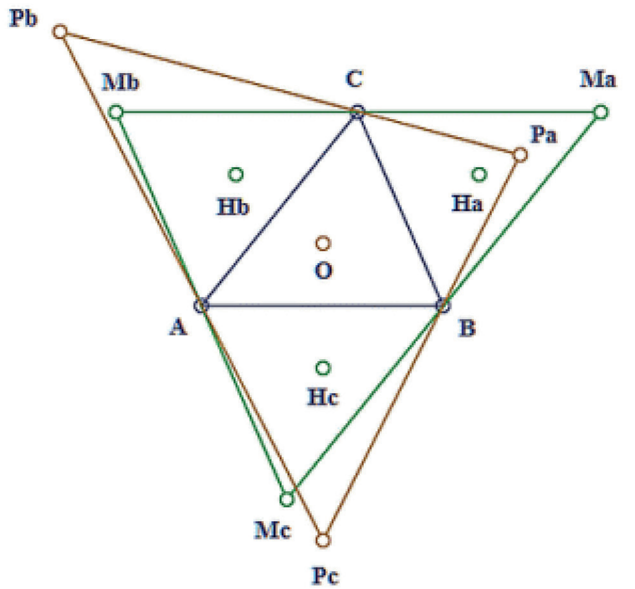

Fig. 3.

In this paper we give proofs of theorems 1 and 2 by using barycentric coordinates. Also, we give examples of antipedal corner products, discovered by the "Discoverer". The enclosed files contain 37 examples of antipedal corner products. Of these, 24 are included in Kimberling and the rest 13 examples are not included in Kimberling. We recommend to the reader to prove the examples of the antipedal corner products, given in the enclosed files.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to (Grozdev and Nenkov, 2012a,b), (Paskalev & Tchobanov, 1985), (Yiu, 2001, edition of 2013), (Douillet, 2012).

We use barycentric coordinates. The reference triangle ABC has vertices $A=(1,0,0)$, $B=(0,1,0)$ and $C=(0,0,1)$. The side lengths of $\triangle ABC$ are denoted by $a=BC$, $b=CA$ and $c=AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is, for $\forall k\in\mathbb{R}\setminus\{0\}$: $P=(u,v,w)$ means that $P\simeq(u,v,w)\simeq(ku,kv,kw)$.

A point $P=(u,v,w)$ is finite, if $u+v+w\neq 0$. A point $P=(u,v,w)$ is normalized, if $u+v+w=1$. A finite point could be put in a normalized form by $P=\left(\frac{u}{s},\frac{v}{s},\frac{w}{s}\right)$, where $s=u+v+w$.

The barycentric coordinates of the antipedal triangle $PaPbPc$, $Pa=(uPa,vPa,wPa)$, $Pb=(uPb,vPb,wPb)$, $Pc=(uPc,vPc,wPc)$ of point $P=(u,v,w)$ are as it follows:

$$uPa=-(2va^2+ua^2+ub^2-uc^2)(2wa^2+uc^2+ua^2-ub^2),$$
$$vPa=(2wa^2+uc^2+ua^2-ub^2)(2ub^2+va^2+vb^2-vc^2),$$
$$wPa=(2va^2+ua^2+ub^2-uc^2)(2uc^2+wc^2+wa^2-wb^2),$$
$$uPb=(2wb^2+vb^2+vc^2-va^2)(2va^2+ua^2+ub^2-uc^2),$$
$$vPb=-(2wb^2+vb^2+vc^2-va^2)(2ub^2+va^2+vb^2-vc^2),$$
$$wPb=(2ub^2+va^2+vb^2-vc^2)(2vc^2+wb^2+wc^2-wa^2),$$
$$uPc=(2vc^2+wb^2+wc^2-wa^2)(2wa^2+uc^2+ua^2-ub^2),$$
$$vPc=(2uc^2+wc^2+wa^2-wb^2)(2wb^2+vb^2+vc^2-va^2),$$
$$wPc=-(2uc^2+wc^2+wa^2-wb^2)(2vc^2+wb^2+wc^2-wa^2).$$

Given two normalized points $P=(u_1,v_1,w_1)$ and $Q=(u_2,v_2,w_2)$, then (Paskalev & Tchobanov, 1985, § 15, Proposition 1):

$$|PQ|^2=-a^2vw-b^2wu-c^2uv, \tag{1}$$

where $u=u_1-u_2$, $v=v_1-v_2$ and $w=w_1-w_2$.

Let DEF be a triangle whose vertices have normalized barycentric coordinates wrt $\triangle ABC$ as it follows: $D=(p_1,q_1,r_1)$, $E=(p_2,q_2,r_2)$ and $F=(p_3,q_3,r_3)$. Let P be a point with normalized barycentric coordinates $P=(p,q,r)$ wrt $\triangle DEF$. Then the barycentric coordinates of $P=(u,v,w)$ wrt $\triangle ABC$ are as it follows (Paskalev & Tchobanov, 1985, § 30):

$$\begin{aligned} u &= p_1 p + p_2 q + p_3 r \\ v &= q_1 p + q_2 q + q_3 r \\ w &= r_1 p + r_2 q + r_3 r \end{aligned} \tag{2}$$

The equation of the line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0 \tag{3}$$

Three lines $p_i x + q_i y + r_i z = 0$, $i = 1,2,3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{4}$$

The intersection of two lines $L_1: p_1 x + q_1 y + r_1 z = 0$ and $L_2: p_2 x + q_2 y + r_2 z = 0$ is the point

$$(q_1 r_2 - q_2 r_1, r_1 p_2 - r_2 p_1, p_1 q_2 - p_2 q_1). \tag{5}$$

Given a point $P=(u,v,w)$, then the complement of P is the point $(v+w, w+u, u+v)$.

3. Proofs of Theorem 1 and 2

Proof of theorem 1. Let *P* be a finite triangle center, $P=(u,v,w)$. By using (1), we find the side lengths $a_1 = |BC| = a$, $b_1 = |PaB|$, and $c_1 = |PaC|$ of $\triangle PaCB$. The barycentric coordinates of the orthocenter *Ha* of $\triangle PaCB$ wrt $\triangle PaCB$ are as it follows:

$Ha = \left(\frac{1}{b_1^2 + c_1^2 - a_1^2}, \frac{1}{c_1^2 + a_1^2 - b_1^2}, \frac{1}{a_1^2 + b_1^2 - c_1^2} \right)$. By using (2), we find the barycentric coordinates of *Ha* wrt $\triangle ABC$ as it follows: $Ha = (-u, u+w, u+v)$. Similarly, we find the barycentric coordinates of *Hb* wrt $\triangle ABC$ as it follows: $Hb = (v+w, -v, u+v)$, and the barycentric coordinates of *Hc* wrt $\triangle ABC$ as it follows: $Hc = (v+w, u+w, -w)$.

Now by using (3) we find the barycentric equations of the lines *AHa*, *BHb* and *CHc* as it follows:

$$AHa\colon\ (u+v)y-(u+w)z=0\,,$$
$$BHb\colon\ (u+v)x-(v+w)z=0\,,$$
$$CHc\colon\ (u+w)x-(v+w)y=0\,.$$

By using (4), we prove that these lines concur in a point. Then, by using (5), we find the point of intersection of the lines *AHa*, *BHb* and *CHc* as the point of intersection R of the lines *AHa* and *BHb* as it follows: $R=(v+w,w+u,u+v)$. Point *R* is the antipedal corner product of point *P* and the Orthocenter. It is easy to see that *R* is the complement of point *P*. This completes the proof.

Proof of theorem 2. Let *P* be a point with barycentric coordinates $P=(u,v,w)$ wrt $\triangle ABC$. The antipedal triangle of the Orthocenter is the antimedial triangle *MaMb-Mc*. Hence, the side lengths of $\triangle MaCB$ are as it follows: $a_1=|BC|=a$, $b_1=|MaB|=b$, and $c_1=|MaC|=c$. The barycentric coordinates of *Pa* wrt $\triangle MaCB$ are as it follows: $Pa=(u,v,w)$. By using (2), we find the barycentric coordinates of *Pa* wrt $\triangle ABC$ as it follows: $Pa=(-u,u+w,u+v)$. Similarly, we find the barycentric coordinates of *Pb* and of *Pc* wrt $\triangle ABC$ as it follows: $Pb=(v+w,-v,u+v)$ and $Pc=(v+w,u+w,-w)$. Points *Pa*, *Pb* and *Pc* coincide with points *Ha*, *Hb* and *Hc* in the proof of theorem 1. Hence the lines *APa*, *BPb* and *CPc* are the same as the lines *AHa*, *BHb* and *CHc* in the proof of theorem 1. This completes the proof.

Supplementary material

The enclosed file “2015-apcp.zip” contains the files quoted in this paper. The reader may download it from the web page of the journal.

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