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COMPLEX SOLUTIONS TO QUADRATIC INEQUALITIES AND THEIR MODELING BY MEANS OF LaTeX

Prof. Dr. Aslanbek Khamidovitch Naziev
Ryazan State University named for S. A. Esenin, Ryazan – Ryazan (Russian Federation)
Ryazan Institute for Education Development – Ryazan (Russian Federation)

Abstract. Many people think that inequalities cannot be considered over the field $\mathbb{C}$ of complex numbers. And really, on $\mathbb{C}$ does not possess an order to make it an ordered field. But other orders exist and can be successfully used for getting answers to some questions which remain unanswered if restricted to the domain of real numbers. The paper notes some discordance in the behavior of the sets of solutions to quadratic equations and inequalities over the field of real numbers. These sets change when the discriminant changes until it is positive, but remain unchanged as soon as discriminant becomes negative. To overcome this oddity, it is suggested to make an exit to the complex plane with some order on it. The consequences of this action are proven, and illustrated by the dynamic model proposed in the work. The model is constructed with the help of LaTeX .

Keywords: ordering complex numbers; reduced square trinomials with real coefficients; real solutions; complex solutions; dynamic model; LaTeX;

1. Introduction

In the behavior of real solutions to quadratic equations and inequalities with real coefficients, there exist some oddities. For example, for inequalities of the form $x^2 + px + q < 0$ with a positive discriminant, the set of solutions is the interval between the roots. The length of this interval decreases as the discriminant decreases, but as soon as the discriminant becomes negative, no matter how much it decreases, nothing changes: just no solutions. In the case of quadratic equations and other quadratic inequalities, the pictures are similar. Transition to the complex plane (Vinberg 2001) and considering it with some order, enables to obtain a more satisfactory picture. And using LaTeX (Znamenskaya 2008) easily allows making this picture dynamic.

2. Main results and their graphical representation

2.1. The order on $\mathbb{R}$

The order on $\mathbb{R}$ may be introduced in different ways, as a primary or secondary notion. In contemporary Russian textbooks on school mathematics, primary is the notion of positive number, while the order is defined on the ground of positivity.

The set of all positive numbers is defined axiomatically by the following axioms:

(Pos-1) Every real number is either positive, or zero, or negative (i. e. opposite to positive).

(Pos-2) No real number is both positive and negative.

(Pos-3) The sum and product of positive numbers are positive numbers.

After that, they give the definition to the notion of order, expressed by the sign '>', reads as '*is greater than*'.

This is made as follows.

Whatever a and b in $\mathbb{R}$, $a > b$ iff $a - b$ is positive.

Taking $b = 0$, we see that $a > 0$ iff a is positive.

As a consequence, we have that

Whatever a and b in $\mathbb{R}$, $a > b$ iff $a - b > 0$. (*)

Now is a good time to caution. Very often the formulation (*) is used as *definition* of the '>' relation. I heard this «definition» from my students, from teachers, from teachers of teachers, and even saw it in a very solid book, many times printed by the publishing house «Vys'shaya Shkola». All the mentioned people do not see that, in the role of definition, this formulation produces a vicious circle; in it '>' is defined through '>'. Being a correct *consequence* of definitions, the formulation (*) is not suitable as a *definition*.

2.2. An order on $\mathbb{C}$

There are many order relations on $\mathbb{C}$. We use a very reasonable one, defined by the agreement

Whatever a and b in $\mathbb{C}$, $a > b$ iff $a - b$ is a positive real number.

Due to this,

Whatever $a \in \mathbb{C}$, $a > 0$ iff $a \in \mathbb{R}$ and $a > 0$ in $\mathbb{R}$.

Consequently,

Whatever a and b in $\mathbb{C}$, $a > b$ iff $a - b \in \mathbb{R}$ and $a - b > 0$ in $\mathbb{R}$.

2.3. Quadratic trinomials with real coefficients over $\mathbb{C}$

In the remaining part of the paper, we consider reduced quadratic trinomials

$$z^2 + pz + q$$

with real coefficients p, q and complex variable z. Represent z in the form $z = x + yi$ with real x and y and picture it as a point (x, y) of coordinate plane Ox. Denote by D the discriminant ($D = p^2 - 4q$) and put for brevity $d = \frac{\sqrt{|D|}}{2}$.

To work with inequalities of the form $f(z)>0$ or $f(z)<0$, we need to know when $f(z)=0$.

PROPOSITION 1. *Whatever* $p,\ q\in\mathbb{R}$ *and* $z\in\mathbb{C}$, $z^2+pz+q\in\mathbb{R} \iff y=0$ or $x=-\dfrac{p}{2}$.

Proof. For all $p,\ q\in\mathbb{R}$ and $z\in\mathbb{C}$,

$$\begin{aligned} z^2+pz+q &= (x+iy)^2+p(x+iy)+q \\ &= x^2+2xyi-y^2+px+pyi+q \\ &= (x^2+px+q-y^2)+iy(2x+p). \end{aligned}$$

So

$$\begin{aligned} z^2+pz+q\in &\iff y(2x+p)=0 \\ &\iff y=0 \text{ or } x=-\frac{p}{2}. \end{aligned}$$

Due to this, all solutions to the equation $z^2+pz+q=0$ and to the inequalities $z^2+pz+q>0$, $z^2+pz+q<0$ belong to the "cross", constituted by the lines $y=0$ and $x=-\frac{p}{2}$ (call them the horizontal and the vertical respectively). This reveals the situation described by the following three propositions (look at the pictures from Figure 1; these pictures correspond to the case $p=q=0$; the general case is presented in the dynamic model placed at the end of the paper).

PROPOSITION 2. *Whatever* $p,\ q\in\mathbb{R}$ *and* $z\in\mathbb{C}$,

$$z^2+pz+q=0:$$

- *at the two points of the horizontal symmetric relative to the vertical and lying at distance from it, — if* $D>0$;
- *at the point of intersection of the horizontal and vertical, — if* $D=0$;
- *at the two points of the vertical symmetric relative to the horizontal and lying at distance* d *from it, — if* $D<0$.

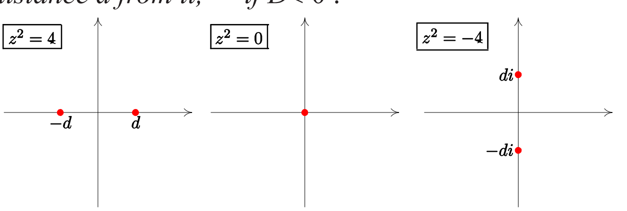

Figure 1

Proof. Let $p,\ q\in\mathbb{R}$ and $z\in\mathbb{C}$. Then, by proposition 1,

$$\begin{aligned} z^2+pz+q=0 &\iff (x^2+px+q-y^2)+iy(2x+p)=0 \\ &\iff \begin{cases} y=0, \\ x^2+px+q=0; \end{cases} \text{ or } \begin{cases} x=-\frac{p}{2}, \\ -\frac{p^2-4q}{4}-y^2=0; \end{cases} \end{aligned}$$

$$\iff \begin{cases} x^2+px+q=0, \\ y=0; \end{cases} \quad \text{or} \quad \begin{cases} x=-\frac{p}{2}, \\ y^2+\frac{D}{4}=0. \end{cases}$$

Now, either $D>0$, or $D=0$, or $D<0$.

- If $D>0$, then the second system (in the last disjunction) has no solutions (because of its second equation), while the first one gives $\begin{cases} x=-\frac{p}{2}\pm d; \\ y=0. \end{cases}$
- If $D=0$, then both systems give $\begin{cases} x=-\frac{p}{2}; \\ y=0. \end{cases}$.
- Lastly, if $D<0$, then the first system has no solutions (because of its first equation), while the second one gives $\begin{cases} x=-\frac{p}{2}; \\ y=\pm d. \end{cases}$

PROPOSITION 3.
Whatever p, $q\in\mathbb{R}$ *and* $z\in\mathbb{C}$, $z^2+pz+q>0$:

- *on the entire horizontal except the interval between roots, — if* $D>0$;
- *on the entire horizontal except the point of its intersection with the vertical,* $D=0$;
- *on the union of the horizontal and the interval between roots on vertical, — if* $D<0$.

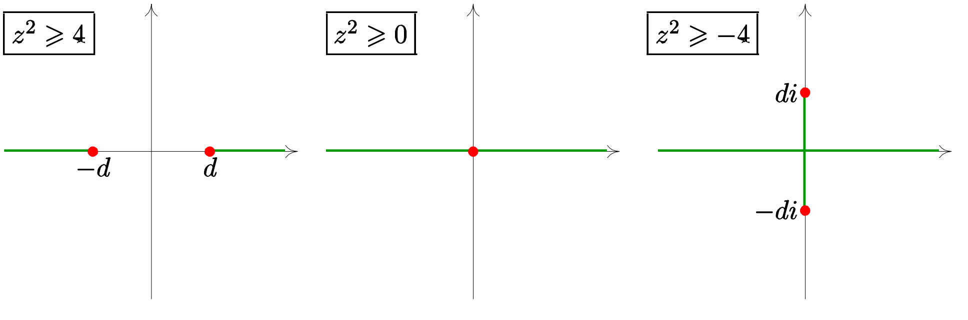

Figure 2

Proof. Let p, $q\in\mathbb{R}$ and $z\in\mathbb{C}$. Then, by Proposition 1,

$$z^2+pz+q>0 \iff (x^2+px+q-y^2)+iy(2x+p)>0$$

$$z^2+pz+q>0 \iff (x^2+px+q-y^2)+iy(2x+p)>0$$
$$\iff \begin{cases} y=0, \\ x^2+px+q>0; \end{cases} \text{ or } \begin{cases} x=-\frac{p}{2}, \\ -\frac{p^2-4q}{4}-y^2>0; \end{cases}$$
$$\iff \begin{cases} x^2+px+q>0, \\ y=0; \end{cases} \text{ or } \begin{cases} x=-\frac{p}{2}, \\ y^2<\frac{-D}{4}. \end{cases}$$

Now, either $D>0$, or $D=0$, or $D<0$.

- If $D>0$, then the second system has no solutions (because of its inequality), while the first one gives $\begin{cases} x\in(-\infty,\ -d)\cup(d,\ +\infty); \\ y=0. \end{cases}$
- If $D=0$, then, again, the second system has no solutions (because of its inequality), while the first one gives $\begin{cases} x\in(-\infty,\ 0)\cup(0,\ +\infty); \\ y=0. \end{cases}$
- Lastly, if $D<0$, then the first system fulfils on the entire horizontal, while the second one gives $\begin{cases} x=-\frac{p}{2}; \\ y\in(-d,\ d). \end{cases}$

Proposition 4. *Whatever* $p,\ q\in\mathbb{R}$ *and* $z\in\mathbb{C}$,:

- *on the union of the vertical and the interval between roots on the horizontal, — if* $D>0$;
- *on the entire vertical except the point of its intersection with the horizontal, — if* $D=0$;
- *on the entire vertical except the interval between roots, — if* $D<0$.

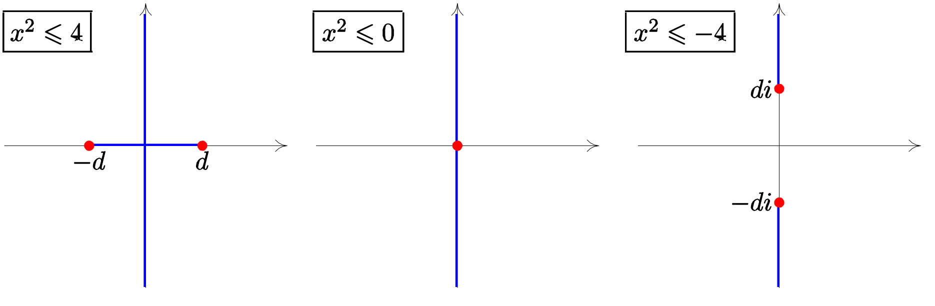

Figure 3

Proof. Let $p,\ q\in\mathbb{R}$ and $z\in\mathbb{C}$. Then, by proposition 1,

$$z^2+pz+q<0 \iff (x^2+px+q-y^2)+iy(2x+p)<0$$

$$\iff \begin{cases} y = 0, \\ x^2 + px + q < 0; \end{cases} \quad \text{or} \quad \begin{cases} x = -\frac{p}{2}, \\ -\frac{p^2-4q}{4} - y^2 < 0; \end{cases}$$

$$\iff \begin{cases} x^2 + px + q < 0, \\ y = 0; \end{cases} \quad \text{or} \quad \begin{cases} x = -\frac{p}{2}, \\ y^2 > \frac{-D}{4}. \end{cases}$$

Now, either $D > 0$, or $D = 0$, or $D < 0$.

- If $D > 0$, than the first system gives $\begin{cases} x \in (-d,\ d), \\ y = 0; \end{cases}$, while the second one gives $\begin{cases} x = -\frac{p}{2}, \\ y \in \mathbb{R}. \end{cases}$

  In the whole it turns out to be that, if $D > 0$, then

$$z^2 + pz + q > 0 \iff \begin{cases} x \in (-d,\ d), \\ y = 0; \end{cases} \quad \text{or} \quad \begin{cases} x = -\frac{p}{2}, \\ y \in \mathbb{R}. \end{cases}$$

- If $D = 0$, then the first system has no solutions (because of its inequality), while the second one gives $\begin{cases} x = -\frac{p}{2}, \\ y \in (-\infty,\ 0) \cup (0,\ +\infty). \end{cases}$

- Lastly, if $D < 0$, then, again, the first system has no solutions (because of its inequality), while the second one gives $\begin{cases} x = -\frac{p}{2}; \\ y \in (-\infty,\ -d) \cup (d,\ +\infty). \end{cases}$

3. The table

Summarize our results in the form of the following table (where $p,\ q \in \mathbb{R}$, $z \in \mathbb{C}$)

The table of main results

	$z^2 + pz + q = 0$	$z^2 + pz + q > 0$	$z^2 + pz + q < 0$
$D > 0$	$z = -\frac{p}{2} \pm d$	$z \in (-\infty, -d) \cup (d, +\infty)$	$z \in (-d, d)$
$D = 0$	$z = -\frac{p}{2}$	$z \in (-\infty, 0) \cup (0, +\infty)$	$z \in -\frac{p}{2} + ((-\infty, 0) \cup (0, +\infty)) \cdot i$
$D < 0$	$z = \pm di$	$z \in \cup(-d,\ d) \cdot i$	$z \in -\frac{p}{2} + ((-\infty,\ -d) \cup (d,\ +\infty)) \cdot i$

4. Exercises

Solve the following equations and inequalities over the set of all complex numbers.

1. $z^2 = 0$; $z^2 > 0$; $z^2 < 0$.

2. $z^2 - 1 = 0$; $z^2 - 1 > 0$; $z^2 - 1 < 0$.

3. $z^2 + 1 = 0$; $z^2 + 1 > 0$; $z^2 + 1 < 0$.

4. $z^2 + z + 1 = 0$; $z^2 + z + 1 > 0$; $z^2 + z + 1 < 0$.

5. $z^2+5z+6=0$; $z^2+5z+6>0$; $z^2+5z+6<0$.

6. $z^2+5z-6=0$; $z^2+5z-6>0$; $z^2+5z-6<0$.

7. $z^2+4z+3=0$; $z^2+4z+3>0$; $z^2+4z+3<0$.

8. $z^2+4z+4=0$; $z^2+4z+4>0$; $z^2+4z+4<0$.

9. $z^2+4z+5=0$; $z^2+4z+5>0$; $z^2+4z+5<0$.

5. Dynamic model

Now, illustrate the described behavior of complex solutions to quadratic equations and inequalities with real coefficients on a dynamic model. This model is made with LaTeX and has the following code.

5.1. The code of model

```
\documentclass{article}
\usepackage{amsmath,amssymb,amsthm}
\usepackage[utf8]{inputenc}
\usepackage[russian]{babel}
\usepackage[dvipsnames,svgnames]{xcolor}
\usepackage[metapost]{mfpic}
\usepackage{multido}
\usepackage[pdftex,designiii,tight,navibar,webpro,russian,unicode]{web}
%
\begin{document}
%
\margins{0pt}{0pt}{0pt}{0pt} % left,right,top, bottom
\screensize{11cm}{11cm} % height, width
\parindent=0pt
\opengraphsfile{quadineq}
\begin{center}
%
\multido{\i=0+1}{400}{%
\begin{mfpic}[100]{-1.55}{1.575}{-1.5}{1.5} %\pen{.8pt}
\gfill[blue]\rect{(-((4-\i/100)**(1/2))/2,-.02),(((4-\i/100)**(1/2))/2,.02)}
\gfill[OliveGreen]\rect{(-1.5,-.02),(-((4-\i/100)**(1/2))/2,.02)}
\gfill[OliveGreen]\rect{(((4-\i/100)**(1/2))/2,-.02),(1.5,.02)}
\pointcolor{red}\point[6pt]{(-((4-\i/100)**(1/2))/2,0),(((4-\i/100)**(1/2))/2,0)}
\gfill[blue]\rect{(-.02,-1.5),(.02,1.5)}
```

```
\pointcolor{red}\point[6pt]{(-((4-\i/100)**(1/2))/2,0),(((4-\i/100)**(1/2))/2,0)}
\tlabel[bl](.5,1){$\boxed{D = 4-\frac{\color{red}\i}{100}}$}
\point[6pt]{(-1.3,1.2)}\tlabel[cc](-.7,1.21){$z^2+pz+q=0$}
\gfill[blue]\rect{(-1.4,.98),(-1.2,1.02)}\tlabel[cc](-.7,1.01){$z^2+pz+q<0$}
\gfill[OliveGreen]\rect{(-1.4,.78),(-1.2,.82)}\tlabel[cc](-.7,.81){$z^2+pz+q>0$}
\xaxis\lines{(0,-1.5),(0,1.5)}\tlabel[tr](-.1,-.1){$-\dfrac{p}{2}$}
\tlabel[br](1.55,.1){$x$}
\end{mfpic}
\newpage}
%
\multido{\i=400+1}{401}{%
\begin{mfpic}[100]{-1.55}{1.575}{-1.5}{1.5} %\pen{.8pt}
\gfill[blue]\rect{(-.02,-1.5),(.02,-(((((\i)/100)-4)**(1/2))/2))}
\gfill[blue]\rect{(-.02,((((\i/100)-4)**(1/2))/2)),(.02,1.5)}
\gfill[OliveGreen]\rect{(-1.5,-.02),(1.5,.02)}
\gfill[OliveGreen]\rect{(-.02,-(((((\i)/100)-
4)**(1/2))/2)),(.02,((((\i/100)-4)**(1/2))/2))}
\pointcolor{red}\point[6pt]{(0,-((((\i/100)-
4)**(1/2))/2)),(0,(((\i/100)-4)**(1/2))/2)}
\tlabel[bl](.5,1){$\boxed{D = 4-\frac{\color{red}\i}{100}}$}
\point[6pt]{(-1.3,1.2)}\tlabel[cc](-.7,1.22){$z^2+pz+q=0$}
\gfill[blue]\rect{(-1.4,.98),(-1.2,1.02)}\tlabel[cc](-.7,1.02){$z^2+pz+q<0$}
\gfill[OliveGreen]\rect{(-1.4,.78),(-1.2,.82)}\tlabel[cc](-.7,.82){$z^2+pz+q>0$}
\xaxis\lines{(0,-1.5),(0,1.5)}\tlabel[tr](-.1,-.1){$-\dfrac{p}{2}$}
\tlabel[br](1.55,.1){$x$}
\end{mfpic}
\newpage}
%
\end{center}
%
\closegraphsfile
%
\end{document}
```

The file with code begins with command `\documentclass{article}` and ends with command `\end{document}`. The part between `\documentclass{article}` and `\begin{document}` is preamble. The code itself is located between the commands `\begin{document}` and `\end{document}`.

The picture is drawn by the package `mfpic` with the option `metapost` (read about the packages mentioned here (Znamenskaya et al., 2008). The screen size is configured using the package web by D. P. Story. Dynamics is provided by using the package `\multido` with the `\newpage` command at the end.

The code consists of two identical parts, each begins with `\multido` and ends with `\newpage}`. The first part produces 401 consecutive slides, which correspond to decreasing of discriminant from 4 to 0 with the step of 1/100. The second part produces 400 consecutive slides, which correspond to decreasing of discriminant from –1/100 to –4 with the same step.

5.2. The model in action

To see the model in action, save to your computer the pdf-file `csqi.pdf` located at http://people.rsu.edu.ru/~a.naziev/Misc/, open it on the page 14, and follow the instructions there.

To produce this model yourself, follow these steps:

1) put the above code in a separate file, for example, `csqi.tex`, and save this file in some directory (preferably — empty);

2) pass the file `csqi.tex` through LaTeX ;

3) after that find in the same directory the file `quadineq.mp` and pass it through `mpost` (in MikTeX it is located in the directory Miktex/miktex/bin);

4) then pass csqi.tex through pdfLaTeX again. As a result, you will receive in the same directory the file csqi.pdf, which is the desired model. Look through this file with the help of Adobe Acrobat Reader in full-screen mode. Do not use any other pdf-reader. Most of them, if not all, reproduced dynamics incorrectly.

6. Conclusion and a look to future

The paper considers only the first, simplest case of studying the complex solutions to inequalities. More complicated cases may be considered, such as the solutions to transcendental inequalities (exponential, logarithmic, etc). This requires some information from complex analysis and, by this reason, does not considered here. The author hopes to turn to these questions in his further publications.

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✉ **Prof. Dr. Aslanbek Khamidivitch Naziev**
Candidate of Mathematical and Physical Sciences
Ryazan State University named after S. A. Esenin,
46, Svobody St.
390048, Ryazan, Russia
Ryazan Institute for Education Development,
2, Firsova St.
390023 Ryazan, Russia
E-mail: a.naziev@365.rsu.edu.ru