https://doi.org/10.53656/math2023-5-1-are

Science in Education
Научно-методически статии

AREAS ASSOCIATED TO A QUADRILATERAL

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Abstract. We study the relationship between the areas of the consecutive quadrilaterals cut from a convex quadrilateral in the plane by means of a finite or infinite number of straight lines intersecting two of its opposite sides. Moreover, we obtain a geometric description of all possible areas obtained in this way given the ratios of the lengths of consecutive segments in which the lines divide these two opposite sides.

Keywords: quadrilateral; area; trihedral angle

1. Introduction

Let $p_1, p_2, \ldots, p_n$ and $p_1', p_2', \ldots, p_n'$ be given n-tuples of positive real numbers. For a convex quadrilateral $ABCD$ in the plane denote by $A_1, A_2, \ldots, A_{n-1}$ and $D_1, D_2, \ldots, D_{n-1}$ the points on the sides AB and CD, respectively such that $AA_1 : A_1A_2 : \ldots : A_{n-1}B = p_1 : p_2 : \ldots : p_n$ and $DD_1 : D_1D_2 : \ldots : D_{n-1}C = p_1' : p_2' : \ldots : p_n'$ (Fig. 1). The purpose of this note is to find a geometric description of the set $\mathcal{A}$ of all possible n-tuples of areas $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, where

$$\mathcal{A}_1 = [AA_1D_1D],\ \mathcal{A}_2 = [A_1A_2D_2D_1],\ \ldots,\ \mathcal{A}_n = [A_{n-1}BCD_{n-1}],$$

when $ABCD$ runs over all convex quadrilaterals in the plane. This problem is motivated by the linear and analytic relations between $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ in the case when $n = 3$ and $p_1 : p_2 : p_3 = p_1' : p_2' : p_3'$ found in (Nikolov 2023).

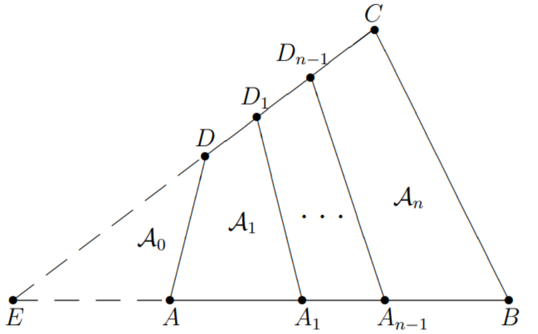

Figure 1

Here we consider the case of arbitrary n and arbitrary ratios (Section 2 and Section 3) as well as that when the sides of the quadrilaterals are divided into infinitely many segments (Section 4). We also show that the last case can be reduced to that when $n = 3$ and $n = 2$ (Section 5).

Understanding the article requires a basic knowledge of linear algebra and analytic geometry.

2. Area attainable points in $\mathbb{R}^3$

We will first consider the case when the sides of the quadrilaterals are divided into 3 segments. Let their ratios be determined by the triples (p_1, p_2, p_3) and (p_1', p_2', p_3').

Definition. A point (x_1, x_2, x_3) in the 3-dimensional Euclidean space $\mathbb{R}^3$ is called *area attainable* if there is a convex quadrilateral *ABCD* in the plane such that $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}) = (x_1, x_2, x_3)$.

It is clear that the set $\mathcal{A}$ of all area attainable points depends on the triples (p_1, p_2, p_3) and (p_1', p_2', p_3'), and is a union of rays in the open first octant $\mathbb{R}_+^3$ with a common initial point $O = (0, 0, 0)$.

Proposition 1. *Set* $\Delta = (p_1 + p_2 + p_3)\, p_1'p_2p_3' - (p_1' + p_2' + p_3')\, p_1'{}_2'p_3$.

a) *If* $\Delta \neq 0$, *then* $\mathcal{A}$ *is the union of two nonintersecting open trihedral angles with a common vertex O and a common face, and an open ray with origin O lying on this face and different from its edges.*

b) *If* $\Delta = 0$, *then* $\mathcal{A}$ *is an open angle with vertex O in the plane*

(1) α: $(p_2p_3' - p_3p_2')\, x_1 + (p_3p_1' - p_1p_3')\, x_2 + (p_1p_2' - p_2p_1')\, x_3 = 0,$

when $p_1 : p_1' \neq p_3 : p_3'$ *or in the plane*

(2) α: $\frac{p_2 + p_3}{p_1} x_1 - \frac{p_1 + 2p_2 + p_3}{p_2} x_2 + \frac{p_1 + p_2}{p_3} x_3 = 0,$

when $p_1 : p_1' = p_3 : p_3'$.

Note that if $\Delta = 0$ *and two of the ratios* $p_1 : p_1'$, $p_2 : p_2'$, $p_3 : p_3'$ *are equal, then the third one is also equal to them.*

Proof. Set $\overrightarrow{v_1} = (p_1, p_2, p_3), \overrightarrow{v_2} = (p_1', p_2', p_3'), \overrightarrow{v_0} = \overrightarrow{v_1} + \overrightarrow{v_2}$,

$\overrightarrow{v_3} = \left(p_1p_1', p_2p_2' + p_2p_1' + p_2'p_1, p_3p_3' + p_3(p_1' + p_2') + p_3'(p_1 + p_2)\right)$,

$\overrightarrow{v_4} = (p_1p_1' + p_1(p_2' + p_3') + p_1'(p_2 + p_3), p_2p_2' + p_2p_3' + p_2'p_3, p_3p_3')$,

and denote by $l_k = \{t\overrightarrow{v_k} : t > 0\}$ the rays determined by these vectors.

Note first that if $AB \parallel CD$, then $l_0 \subset \mathcal{A}$. Let now $AB \nparallel CD$ and denote by Q_1 the set of convex quadrilaterals *ABCD* in the plane such that *A* lies between *B* and $E = AB \cap CD$ (the set Q_2 is defined analogously when *B* lies between *A* and *E*). Set $EA : AA_1 = p_0 : p_1$, $ED : DD_1 = p_0' : p_1'$, $\mathcal{A}_0 = [EAB] = sp_0p_0'$. Then

$$\frac{\mathcal{A}_0+\mathcal{A}_1}{\mathcal{A}_0}=\frac{(p_0+p_1)(p_0'+p_1')}{p_0p_0'},$$
$$\frac{\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2}{\mathcal{A}_0}=\frac{(p_0+p_1+p_2)(p_0'+p_1'+p_2')}{p_0p_0'},$$
$$\frac{\mathcal{A}_0+\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3}{S_0}=\frac{(p_0+p_1+p_2+p_3)(p_0'+p_1'+p_2'+p_3')}{p_0p_0'},$$

from where we get

$$\mathcal{A}_1=s(p_1p_0'+p_1'p_0+p_1p_1'),$$
$$\mathcal{A}_2=s(p_2p_0'+p_2'p_0+p_2p_2'+p_2p_1'+p_2'p_1),$$
$$\mathcal{A}_3=s\big(p_3p_0'+p_3'p_0+p_3p_3'+p_3(p_1'+p_2')+p_3'(p_1+p_2)\big).$$

Let M_1 be the 3×3 matrix with rows $\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3}$. Then the identities above may be written in the following matrix form $(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)=(sp_0',sp_0,s)M_1$. Since the map $ABCD\to(sp_0',sp_0,s)$ is a surjection from Q_1 onto the open first octant $\mathbb{R}_+{}^3$ of $\mathbb{R}^3$ (take, for example, $\angle AED=90°$, $AE=2p_0$ and $E=sp_0'$), it follows that the set $\mathcal{A}'$ of area attainable points for which *A* lies between *E* and *B* is the image of $\mathbb{R}_+{}^3$ under the linear transformation M_1 of $\mathbb{R}^3$ with matrix M_1.

Analogously, the set $\mathcal{A}''$ of area attainable points for which *B* lies between *A* and *E* is the image of $\mathbb{R}_+{}^3$ under the linear transformation M_2 of $\mathbb{R}^3$ with matrix M_2 whose rows are $\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_4}$.

It is clear that the set $\mathcal{A}$ of all area attainable points is given by $\mathcal{A}=l_0\cup\mathcal{A}'\cup\mathcal{A}''$ and now we will describe geometrically this set depending on whether $\Delta\neq 0$ or $\Delta=0$, i.e. if $\operatorname{rank} M_1=\operatorname{rank} M_2$ is equal to 3 or 2 (the ranks of M_1 and M_2 are never equal to 1.)

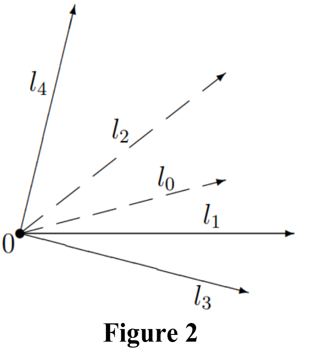

Figure 2

Let $\Delta\neq 0$. Then $\mathcal{A}'$ and $\mathcal{A}''$ are the open trihedral angles with edges the rays l_1,l_2,l_3 and l_1,l_2,l_4 (the images of the positive axes under the linear transformations M_1 and M_2, respectively), and l_0 lies on the face with edges l_1 and l_2 (Fig. 2). It is easy to check that the scalar triple product of the vectors $\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3}$ is equal to $\det M_1=-\Delta$, and that of $\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_4}$ is equal to $\det M_2=-\Delta$. In particular, these two triples of vectors determine opposite orientations on $\mathbb{R}^3$ and therefore $\mathcal{A}'\cap\mathcal{A}''=\emptyset$. Note that this follows also from the identity:

$$v_3+\overrightarrow{v_4}=(p_1'+p_2'+p_3')\overrightarrow{v_1}+(p_1+p_2+p_3)\overrightarrow{v_2}.$$

Let now $\Delta = 0$. Then $\det M_1 = \det M_2 = 0$ and the rays l_0, l_1, l_2, l_3, l_4 lie on a plane $\alpha \supset \mathcal{A}$. It is defined by (1) or (2), which follows respectively from $\alpha \perp \overrightarrow{v_1} \times \overrightarrow{v_2}$ or $\alpha \perp \overrightarrow{v_3} \times \overrightarrow{v_4}$.

Note that the image $M_1(\mathbb{R}_+^3)$ (resp. $M_2(\mathbb{R}_+^3)$) consists of the points in $\mathbb{R}^3$ whose vectors are linear combinations of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ (resp. $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_4}$) with positive coefficients. A direct check shows that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are linear combinations of $\overrightarrow{v_3}$ and $\overrightarrow{v_4}$ with positive coefficients. Hence in the case $p_1 : p_1' \neq p_3 : p_3'$, we have $\mathcal{A}' = \angle(l_1, l_3)$, $\mathcal{A}'' = \angle(l_2, l_4)$ if $l_2 \subset \angle(l_1, l_3)$ and $\mathcal{A}' = \angle(l_1, l_3)$, $\mathcal{A}'' = \angle(l_1, l_4)$ if $l_1 \subset \angle(l_2, l_3)$; in particular, $l_0 \subset (l_1, l_2) = \mathcal{A}' \cap \mathcal{A}''$. In the case $p_1 : p_1' = p_3 : p_3'$ it follows that $\mathcal{A}' = \angle(l_0, l_3)$ and $\mathcal{A}'' = \angle(l_0, l_4)$. Hence in both cases $\mathcal{A} = \angle(l_3, l_4)$.

The above arguments imply the following:

Corollary 2. *If* $AB \| CD$, *then*

(3) $$\mathcal{A}_1 : \mathcal{A}_2 : \mathcal{A}_3 = (p_1 + p_1') : (p_2 + p_2') : (p_3 + p_3').$$

Conversely, *if* $\Delta \neq 0$ and (3) *is fulfilled, then* $AB \| CD$.

Remark. a) If $p_1 : p_2 : p_3 = p_1' : p_2' : p_3'$, then $AB \| CD$ follows from the weaker assumptions $\mathcal{A}_1 : \mathcal{A}_2 = p_1 : p_2$, $\mathcal{A}_2 : \mathcal{A}_3 = p_2 : p_3$ or $\mathcal{A}_3 : \mathcal{A}_1 = p_3 : p_1$.

b) If $\Delta = 0$, but $p_1 : p_1' \neq p_3 : p_3'$, then there are quadrilaterals from Q_1 and Q_2, such that (3) is fulfilled.

The proof of Proposition 1 implies also an analytic description of the area attainable points $\mathcal{A}$. For example, we have the following:

Corollary 3. (Nikolov 2023, Proposition 3) *If* $p_1 : p_2 : p_3 = p_1' : p_2' : p_3'$, *then the area attainable points* (x_1, x_2, x_3) *are those for which (2) is fulfilled*, $x_1 > 0$ *and*

(4) $$\frac{p_3^2}{p_1(p_1+2p_2+2p_3)} < \frac{x_3}{x_1} < \frac{p_3(2p_1+2p_2+p_3)}{p_1^2}.$$

Proof. Note that

$$l_3 = \{t\left(p_1^2, p_2(p_2 + 2p_1), p_3(2p_1 + 2p_2 + p_3)\right) : t > 0\},$$

$$l_4 = \{t(p_1(p_1 + 2p_2 + 2p_3), p_2(p_2 + 2p_3), p_3^2) : t > 0\}$$

and it follows from the proof of Proposition 1 that the area attainable points (x_1, x_2, x_3) are those for which (2) is fulfilled, and

$$x_1 = \lambda p_1^2 + \mu p_1(p_1 + 2p_2 + 2p_3), \; x_3 = \lambda p_3(2p_1 + 2p_2 + p_3) + \mu p_3^2, \; \lambda, \mu > 0.$$

Solving this system with respect to λ and μ we see that it is equivalent to $x_1 > 0$ and (4).

3. Area attainable points in $\mathbb{R}^n$

In this section we will obtain a geometric description of the set $\mathcal{A}$ of area attainable points when the sides AB and CD of a convex quadrilateral $ABCD$ in the plane are divided into n segments. In this case $\mathcal{A}$ is a subset of the n-dimensional Euclidean space $\mathbb{R}^n$.

Let $(p_1, \dots, p_n)$ and $(p'_1, \dots, p'_n)$ be two n-tuples of real numbers which determine the ratios of the consecutive n segments of which the sides AB and CD are divided. The case when these two n-tuples coincide has been considered in (Nikolov 2023), where an analytic description of the set $\mathcal{A}$ is given.

If $n = 2$, then the proof of Proposition (1) for $p_3 = p'_3 = 0$ implies that $\mathcal{A}$ is an open angle with vertex O and arms

$$\{t(p_1p'_1, p_2p'_2 + p_2p'_1 + p'_2p_1): t > 0\},\ \{t(p_1p'_1 + p_1p'_2 + p'_1p_2, p_2p'_2): t > 0\}.$$

The analogue of Proposition (1) in higher dimensions is the following:

Proposition 4. *Let* $n \geq 4$ *and set*:

$$\Delta_i = (p_{i-1} + p_i + p_{i+1})p'_{i-1}p'_{i+1}p_i - (p'_{i-1} + p'_i + p'_{i+1})p_{i-1}p_{i+1}p'_i,\ 2 \leq i \leq n-1.$$

a) If at least one of the numbers Δ_i *is different from* 0, *then* $\mathcal{A}$ *is the union of two nonintersecting open trihedral angles in the same* 3*-dimensional vector subspace of* $\mathbb{R}^n$ *with a common vertex* $O = (0,0,\dots,0)$ *and a common face, and an open ray with origin* O *lying in this face and different from its edges*;

b) If all numbers Δ_i *are equal to* 0 *then* $\mathcal{A}$ *is an open angle with vertex* O.

Sketch of proof. Set $\overrightarrow{v_1} = (p_1, \dots, p_n)$, $\overrightarrow{v_2} = (p'_1, \dots, p'_n)$, $\overrightarrow{v_0} = \overrightarrow{v_1} + \overrightarrow{v_2}$, $\overrightarrow{v_3} = (s_1, \dots, s_n)$, $\overrightarrow{v_4} = (t_1, \dots, t_n)$, where

$$s_i = -p_ip'_i + p_i\sum_{j=1}^{i} p'_j + p'_i\sum_{j=1}^{i} p_j,\ t_i = -p_ip'_i + p_i\sum_{j=i}^{n} p'_j + p'_i\sum_{j=i}^{n} p_j,\ 1 \leq i \leq n.$$

Using the above notations and those in Section 1 we see that if $AB \nparallel CD$, then the areas $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are given by the following formulas: $\mathcal{A}_i = s(p_0p'_i + p'_0p_i + s_i), 1 \leq i \leq n$ or $\mathcal{A}_i = s(p_0p'_i + p'_0p_i + t_i), 1 \leq i \leq n$, depending on whether A lies between $E = AB \cap CD$ and B or B lies between A and E.

Denote by $l_j = \{t\overrightarrow{v_j}: t > 0\}$ the rays determined by the vectors $v_j, 0 \leq j \leq 4$.

In case b) the set $\mathcal{A}$ is an open angle with vertex O and arms l_3 and l_4, which determine a plane α in $\mathbb{R}^n$. Let us note that in this case either $p_j : p_k \neq p'_j : p'_k f\ or\ j \neq k$, or $p_1 : p_2 : \dots : p_n = p'_1 : p'_2 : \dots : p'_n$. To see this we set $q_i = p_i/p_{i-1}, q'_i = p'_i/p'_{i-1}$ and rewrite the identities $\Delta_i = 0$ in the form

$$q_{i+1} - q'_{i+1} = \frac{(1 + q'_{i+1})q_{i+1}}{(1 + q_i)}(q_i - q'_i).$$

It follows that if $q_2 \neq q'_2$, then the sequence with general term p_{i+1}/p'_{i+1} is strictly monotonic and therefore $p_j : p_k \neq p'_j : p'_k$ for $j \neq k$. If $q_2 = q'_2$ then $q_{i+1} = q'_{i+1}$, i.e. $p_1 : p_2 : \dots : p_n = p'_1 : p'_2 : \dots : p'_n$.

Let us note also that the plane α is the intersection of the linearly independent hyperplanes $\alpha_i, 2 \leq i \leq n-1$, defined by the equations:

(5) $\alpha_i: (p_i p'_{i+1} - p_{i+1} p'_i) x_{i-1} + (p_{i+1} p'_{i-1} - p_{i-1} p'_{i+1}) x_i + (p_{i-1} p'_i - p_i p'_{i-1}) x_{i+1} = 0$

in case a) and

(6) $$\alpha_i: \frac{p_{i+1}+p_i}{p_{i-1}} x_{i-1} - \frac{p_{i-1}+2p_i+p_{i+1}}{p_i} x_i + \frac{p_{i-1}+p_i}{p_{i+1}} x_{i+1} = 0$$

in case b).

In case a) we have $\mathcal{A} = l_0 \cup \mathcal{A}' \cup \mathcal{A}''$, where $\mathcal{A}'$ is a trihedral angle with edges l_1, l_2, l_3, and $\mathcal{A}''$ – that with edges l_1, l_2, l_4. The identity $\overrightarrow{v_3} + \overrightarrow{v_4} = \overrightarrow{v_1} \sum_{j=1}^{n} p'_j + \overrightarrow{v_2} \sum_{j=1}^{n} p_j$ shows that these two trihedral angles lie in the same 3-dimensional vector subspace of $\mathbb{R}^n$ and l_3, and l_4 lie on different sides of the plane determined by l_1 and l_2.

This 3-dimensional vector subspace can be described as the intersection of $n-3$ linearly independent hyperplanes β_i. For example, if $\Delta_k \neq 0$ for some k we may choose

(7) β_i: $x_i + c_{k-1,i} x_{k-1} + c_{k,i} x_k + c_{k+1,i} x_{k+1} = 0,\ i = 1, \dots, k-2, k+2, \dots, n,$

where the coefficients $c_{k-1,i}, c_{k,i}, c_{k+1,i}$ are the solutions x, y, z of the system:

$$p_{k-1}x + p_k y + p_{k+1} z = -p_i$$
$$p'_{k-1}x + p'_k y + p'_{k+1} z = -p'_i$$
$$s_{k-1}x + s_k y + s_{k+1} z = -s_i.$$

The explicit formulas for these solutions are given by Kramer's formulas.

4. Area attainable points in ℓ^1

Now we will consider the case when the sides of the quadrilaterals are divided into infinitely many segments.

Let $p = (p_i)_{i \in \mathbf{N}}$ and $p' = (p'_i)_{i \in \mathbf{N}}$ be two infinite sequences of positive integers. We will say that an infinite sequence $x = (x_i)_{i \in \mathbf{N}}$ of real numbers is area attainable if there is a quadrilateral $ABCD$ in the plane and points $A_0 = A, A_1, A_2, \dots$ and $D_0 = D, D_1, D_2, \dots$ on the sides AB and DC, such that $A_0A_1 : A_1A_2 : \dots = p_1 : p_2 : \dots$, $D_0D_1 : D_1D_2 : \dots = p'_1 : p'_2 : \dots$ and $x_i = \mathcal{A}_i = [A_{i-1}A_iD_iD_{i-1}]$ for all $i \in \mathbf{N}$.

Denote by ℓ^1 the space of all sequences $r = (r_i)_{i \in \mathbf{N}}$ of real numbers such that $\sum_{i=1}^{\infty} |r_i| < \infty$. It is clear that the set $\mathcal{A}$ of area attainable points is empty if $p \notin \ell^1$ or $p' \notin \ell^1$.

Proposition 5. *Let $p, p' \in \ell^1$ be given infinite sequences of positive real numbers and let Δ_i $(i \geq 2)$ be the numbers defined in Proposition 4.*

a) *If at least one of the numbers Δ_i is different from* 0, *then $\mathcal{A}$ is the union of two nonintersecting open trihedral angles in the same* 3*-dimensional vector* *subspace of* ℓ^1 *with a common vertex* $O = (0, 0, \dots)$ *and a common face, and an open ray with origin* O *lying on this face and different from its edges.*

b) *If all numbers* Δ_i *are equal to* 0, *then* $\mathcal{A}$ *is an open (2-dimensional) angle in* ℓ^1 *with vertex* O.

The proof of this proposition is the same as that of Proposition 4 with some obvious changes. For example, in the description of the set $\mathcal{A}''$ we may assume that $A_n \to B, D_n \to C$ and then the identity:

$$\frac{\sum_{j=i}^{\infty} \mathcal{A}_j}{\mathcal{A}_0} = \frac{\left(p_0 + \sum_{j=i}^{\infty} p_j\right)\left(p_0' + \sum_{j=i}^{\infty} p_j'\right)}{p_0 p_0'}$$

implies that in this case $\mathcal{A}_i = s(p_0 p_i' + p_0' p_i + t_i)$, where $t_i = -p_i p_i' + \sum_{j=i}^{\infty} p_j' \, p_i + p_i' \sum_{j=i}^{\infty} p_j \, , i \notin \mathbb{N}$.

Then $\overrightarrow{v_3}, \overrightarrow{v_4} \notin \ell^1$ since $\sum_{i=1}^{\infty}(s_i + t_i) = 2\sum_{i=1}^{\infty} p_i \sum_{i=1}^{\infty} p_i'$.

The three and two dimensional vector subspaces of ℓ^1 containing the set $\mathcal{A}$ of area attainable sequences can be defined as intersections of countably many linearly independent hyperplanes in ℓ^1. Note also that Proposition 4 is a particular case of Proposition 5 in view of the natural embedding of $\mathbb{R}^n$ in ℓ^1.

5. Back to $\mathbb{R}^3$ and $\mathbb{R}^2$

The considerations in previous sections allow us to reduce the case of ℓ^1 (in particular, $\mathbb{R}^n$) to the cases $\mathbb{R}^3$ or $\mathbb{R}^2$.

Proposition 6. Let $p, p' \in \ell^1$.

a) *If* $\Delta_k \neq 0$ *for some* $k \geq 2$, *then* $x \in \mathcal{A}'$ *precisely when* (7) *is fulfilled for all* $i \neq k-1, k, k+1$ *and* $\left(\sum_{j=1}^{k-1} x_j \, , x_k, x_{k+1}\right)$ *is an area attainable point in* $\mathbb{R}^3$ *corresponding to the triples* $\left(\sum_{j=1}^{k-1} p_j \, , p_k, p_{k+1}\right)$ *and* $\left(\sum_{j=1}^{k-1} p_j' \, , p_k', p_{k+1}'\right)$, *such that A is between B and E.*

Analogously, $x \in \mathcal{A}''$ *precisely when* (7) *is fulfilled for all* $i \neq k-1, k, k+1$ *and* $\left(x_{k-1}, x_k, \sum_{j=k+1}^{\infty} x_j\right)$ *is an area attainable point in* $\mathbb{R}^3$ *corresponding to the triples* $\left(p_{k-1}, p_k, \sum_{j=k+1}^{\infty} p_j\right)$ *and* $\left(p_{k-1}', p_k', \sum_{j=k+1}^{\infty} p_j'\right)$, *and such that B is between A and E.*

b) *If* $\Delta_k = 0$ *for all* $k \geq 2$, *then* $x \in \mathcal{A}$ *precisely when* $x_1 > 0$, (5) *or* (6) *is fulfilled for all* $i \geq 2$ and

$$\frac{\Sigma_2}{\Sigma_1} < \frac{x_2}{x_1} < \frac{p_2 p_2' + p_2 p_1' + p_2' p_1}{p_1 p_1'},$$

where

$$\Sigma_1 = p_1 p_1' + p_1 \sum_{j=2}^{\infty} p_j' + p_1' \sum_{j=2}^{\infty} p_j \,, \Sigma_2 = p_2 p_2' + p_2 \sum_{j=3}^{\infty} p_j' + p_2' \sum_{j=3}^{\infty} p_j .$$

We leave the proof to the readers noting that case b) is similar to that in Corollary 3.

In case b) we have

$$\begin{vmatrix} p_1 & p_1 & p_{i+1} \\ p_1' & p_2' & p_{i+1}' \\ x_1 & x_2 & x_{i+1} \end{vmatrix} = 0$$

and if $p_1 : p_2 : \ldots \neq p_1' : p_2' : \ldots$, then

$$x_{i+1} = \frac{x_1(p_2 p_{i+1}' - p_{i+1} p_2') + x_2(p_{i+1} p_1' - p_1 p_{i+1})}{p_2 p_1' - p_1 p_2'}, \quad i \in \mathbf{N}.$$

Set $\widetilde{x_i} = x_i / p_i, i \in \mathbf{N}$ and

$$s_i = \frac{p_1 + p_i}{p_1 + p_2} + \frac{2}{p_1 + p_2} \sum_{j=2}^{i-1} p_j \,, \quad i \geq 3.$$

Corollary 7. *If* $p = p' \in \ell^1$ *then a sequence of positive real numbers* $x = (x_i)_{i \in \mathbb{N}}$ *is area attainable if and only if* $x_1 > 0$,

(8) $$x_i = \widetilde{x_1} + s_i(\widetilde{x_2} - \widetilde{x_1}), \quad i \geq 3$$

and

$$1 - \frac{p_1 + p_2}{p_1 + 2\sum_{j=2}^{\infty} p_j} < \frac{\widetilde{x_2}}{\widetilde{x_1}} < 2 + \frac{p_2}{p_1}.$$

Proof. It is enough to check (8). For that purpose, we rewrite (6) in the form

$$\frac{\widetilde{x_{j+1}} - \widetilde{x_j}}{p_{j+1} + p_j} = \frac{\widetilde{x_j} - \widetilde{x_{j-1}}}{p_j + p_{j-1}}.$$

Then

$$\widetilde{x_{j+1}} - \widetilde{x_j} = (p_{j+1} + p_j) \frac{\widetilde{x_2} - \widetilde{x_1}}{p_2 + p_1}$$

and it remains to sum up these identities for $j = 2, \ldots, i-1$.

Note that for $p = p' \in \mathbb{N}$ the above corollary is a more compact expression of Proposition 4 in (Nikolov 2023).

REFERENCES

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