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APOLLONIAN SPHERE AND PROPERTIES OF STEREOGRAPHIC PROJECTION AROUND THE LEMOINE POINT

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Abstract. The definition and some of the properties of the Apollonian circle in the plane find their analogies in the Euclidean three-dimensional space. Thus, we manage to introduce a new concept in solid geometry that we call an "Apollonian sphere". It appears that the Apollonian sphere not only possesses classical properties similar to the Apollonian circle such as orthogonality and coaxiality, but also analogies of its lesser-known connection with the Lemoine point and the circumcenter. We also discover two notable properties of stereographic projection that we prove with an Apollonian sphere. They include collinearity of the projection point with the Lemoine points of the projection and the projected triangles or with the centers of their Apollonian circles. Moreover, we connect the newly introduced concepts and the rich configurations they generate with Olympiad geometry.

Keywords: Apollonian sphere; stereographic projection; harmonic tetrahedron; Lemoine point; center of Apollonian circle; Olympiad geometry

1. Introduction

The first goal of the paper is to look for and investigate the properties of the 3D space version of the Apollonian circle that we name an "Apollonian sphere". The second goal of the article is to search for applications of the Apollonian sphere in 3D space, which brings us to the discovery of two notable properties of stereographic projection. The third goal of the paper is to find applications of the Apollonian sphere and the newly discovered properties of stereographic projection in Olympiad geometry.

In Section 2, we define the Apollonian sphere, and we look at its properties. In Section 3, we prove two new properties of stereographic projection via an Apollonian sphere, constituting of collinearity of notable points (Lemoine points/centers of Apollonian circles) from the projection and the projected triangles with the projection point. Finally, in Section 4, we examine the applications of the new concepts and configurations in Olympiad geometry. We formulate an Olympiad problem based on the discovered properties of stereographic projection, and we provide an original solution to another one via an Apollonian sphere[1].

2. Apollonian Sphere

The familiar properties of the Apollonian circle in the plane such as orthogonality and coaxiality, (Andreescu et al. 2016, pp. 275 – 279), find their analogies in 3D space. We call the resulting figure an "Apollonian sphere". In stereometry, however, we investigate more in-depth connections with notable points, lines, and planes, which we discuss in Section 2.

Theorem 2.1. Let $\triangle ABC$ be a scalene triangle. Then the locus of points X, satisfying $\frac{XA}{XB} = \frac{CA}{CB}$, is a sphere Σ_C, which we will call the *Apollonian sphere* for the triangle corresponding to the vertex C.

Proof. 1) Existence. Let CL_1 and CL_2 be the interior and the exterior angle bisector of $\angle ACB$ in $\triangle ABC$ (see fig. 1). Let M be the midpoint of L_1L_2, and let us construct the sphere $\Sigma_C(M, r)$ where

$$r = MC = ML_1 = ML_2.$$

Let point $X \in \Sigma_C$, and let us construct the plane passing through point X and the line AB. We note that

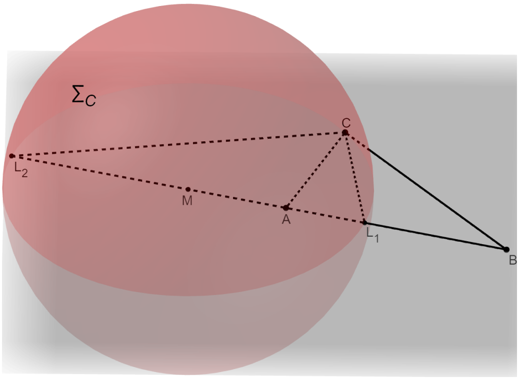

Figure 1

$$\frac{BL_1}{L_1A} : \frac{BL_2}{L_2A} = \frac{BC}{CA} : \frac{BC}{CA} = 1 \Rightarrow (B, L_1, A, L_2) = 1.$$

But $\angle L_2XL_1 = 90^\circ$ (L_1L_2 – diameter in Σ_C). From the properties of the harmonic division, (Boev 2010, p. 80), it follows that XL_1 is an angle bisector of $\angle AXB \Rightarrow \dfrac{XA}{XB} = \dfrac{AL_1}{L_1B} = \dfrac{AC}{BC}$.

2) Uniqueness. Let point Y satisfy $\dfrac{YA}{YB} = \dfrac{AC}{BC}$. Point Y and the line AB form a plane. But $\dfrac{AC}{BC} = \dfrac{AL_1}{BL_1} \Rightarrow \dfrac{YA}{YB} = \dfrac{AL_1}{BL_1} \Rightarrow YL_1$ is an angle bisector of $\angle AYB$, and $(B, L_1, A, L_2) = 1$. From the properties of the harmonic division, (Boev 2010, p. 80), it follows that $\angle L_1YL_2 = 90^\circ \Rightarrow Y \in \Sigma_C(M, r = ML_1)$. □

We will prove an analogue in 3D space of the coaxiality of the Apollonian circle.

Theorem 2.2. The three Apollonian spheres for a given scalene $\triangle ABC$ intersect at a circle.

Proof. Let us construct the spheres Σ_C and Σ_B (see fig. 2). They are intersecting since the corresponding Apollonian circles are intersecting, and they are great circles in the Apollonian spheres. Let $\Sigma_C \cap \Sigma_B = \sigma$, and let point $X \in \sigma \overset{2.1.}{\Longrightarrow}$

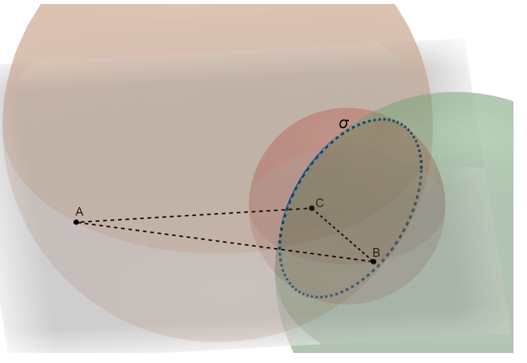

Figure 2

$$\frac{XA}{XB} = \frac{CA}{CB} \quad \cap \quad \frac{XC}{XA} = \frac{BC}{BA}.$$

If we multiply the two equations, we get that $\dfrac{XC}{XB} = \dfrac{AC}{AB}$. By Theorem 2.1., it follows that point $X \in \Sigma_A$. The same applies for each point on σ. □

Note. As the centers of the three Apollonian spheres coincide with the centers of the three Apollonian circles, we know from planimetry that the centers lie on a line (the Lemoine axis).

The next statement is the stereometric equivalent of the orthogonality of the Apollonian circle.

Theorem 2.3. The Apollonian spheres for a given scalene $\triangle ABC$ are orthogonal to every sphere passing through points A, B, and C.

Proof. Firstly, we will introduce the following fundamental Lemma:

Lemma: Let point M lie on a circle k that lies on a sphere Σ. Let t be the tangent line to k at M, and let π_M be the tangent plane to Σ at M. Then the line t lies on the plane π_M.

Back to our main problem: let us take an arbitrary sphere Σ passing through points A, B, and C (see fig. 3). We will prove that $\Sigma_C \perp \Sigma$ (analogously for the other two Apollonian spheres). It is clear that the spheres Σ_C and Σ are intersecting. Let $\Sigma_C \cap \Sigma = \omega$, and let us arbitrarily take point $X \in \omega$. As $X \in \Sigma_C$, by Theorem 2.1., $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ (∗).

Let us define the points as in Theorem 2.1. Point X and the line AB determine a plane ⇒

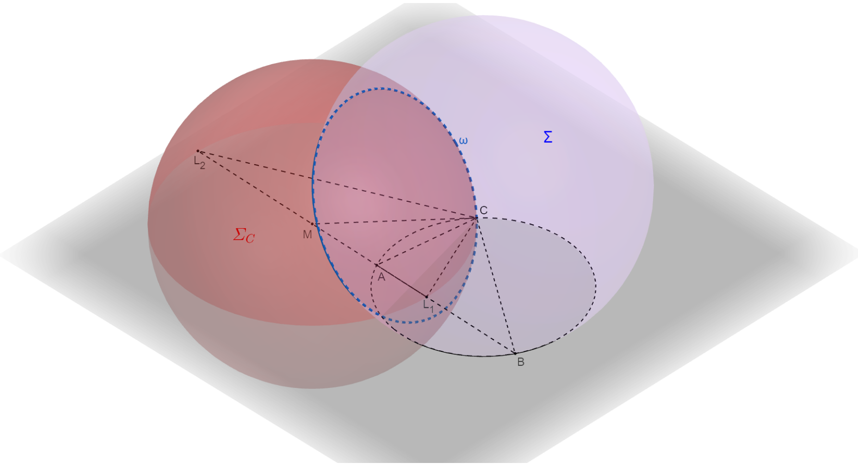

Figure 3

$$\frac{XA}{XB} \overset{(*)}{=} \frac{CA}{CB} = \frac{AL_1}{L_1B} \Rightarrow XL_1 \text{ is an angle bisector of } \angle AXB.$$

Let us denote $\angle ABX$ by φ, and $\angle AXB$ by 2ψ. It is clear that we have the equalities $\angle AXL_1 = \angle BXL_1 = \psi$. Therefore,

$$\angle MXL_1 = \angle MXA + \angle AXL_1 = \angle MXA + \psi.$$

But as an exterior angle for $\triangle XBL_1$,

$$\angle ML_1X = \angle L_1XB + \angle L_1BX = \psi + \varphi.$$

It remains to be considered that since $MX = ML_1$, then we have the equality $\angle MXL_1 = \angle ML_1X \Rightarrow$

$$\angle MXA + \psi = \psi + \varphi \Leftrightarrow \angle MXA = \varphi.$$

Therefore, MX is the tangent line for the circle $k[ABX]$, and $k[ABX] \in \Sigma$. Applying the Lemma, it follows that the tangent plane π_X to Σ at point X passes through the center of Σ_C, point M. Similarly, the result applies for every point $Y \in \omega$: the tangent plane π_Y to Σ at point Y passes through the center of Σ_C, point $M \Rightarrow \Sigma_C \perp \Sigma$. □

The next theorem is a 3D equivalent of the connection of the Apollonian circle with the notable points of the triangle. In this stereometric configuration, we prove the existence of new notable elements, and we examine their

Theorem 2.4. Let $\triangle ABC$ be a scalene triangle with circumcenter O, Lemoine point L, and Apollonian spheres $\Sigma_A(M), \Sigma_B(N)$, and $\Sigma_C(P)$. Let the sphere Σ pass through points A, B, and C. Let Σ intersect Σ_A, Σ_B, and Σ_C at the circles k', k'', and k''', lying on the planes α, β, and γ, respectively. Let us denote the line through the points M, N, and P by s. Then the following facts apply:

a) $\alpha \cap \beta \cap \gamma =$ line q; b) $L \in q$; c) $s \perp q$;

d) The orthogonal projection of line q onto the plane through $\triangle ABC$ passes through O.

Proof.

a) The orthogonality in Theorem 2.3. leads to the polar reciprocation, (Prasolov 2010b, pp. 51 – 53), between the centers of the Apollonian spheres and their radical planes with Σ with respect to Σ (see fig. 4). Therefore, $\pi_\Sigma(M) = \alpha$, $\pi_\Sigma(N) = \beta$, and $\pi_\Sigma(P) = \gamma$. But points M, N, and P are collinear. Thus, $\alpha \cap \beta \cap \gamma =$ line q. □

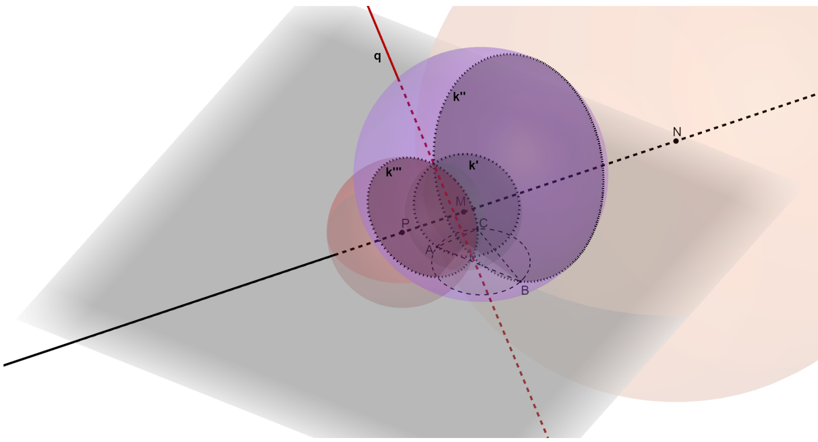

Figure 4

Corollary. The lines s and q are polar (Smith 1893, p. 40).

b) Without loss of generality, let us take the circles k' and k'' (fig. 5).

Circles k' and k (the circumcircle of $\triangle ABC$) lie on a sphere (Σ) and have a common point (A), and as it is clear that they do not coincide, then they have a second common point, point A_1. But $A_1 \in k' \in \Sigma_A$, and A_1 lies on the plane through $\triangle ABC$. Therefore, A_1 lies on the Apollonian circle corresponding to the vertex A. Therefore, from planimetry, (Andreescu et al. 2016, p. 277), we know that AA_1 is a symmedian in $\triangle ABC$.

Circles k'' and k lie on a sphere (Σ) and have a common point (B), and as it is clear that they do not coincide, then they have a second common point,

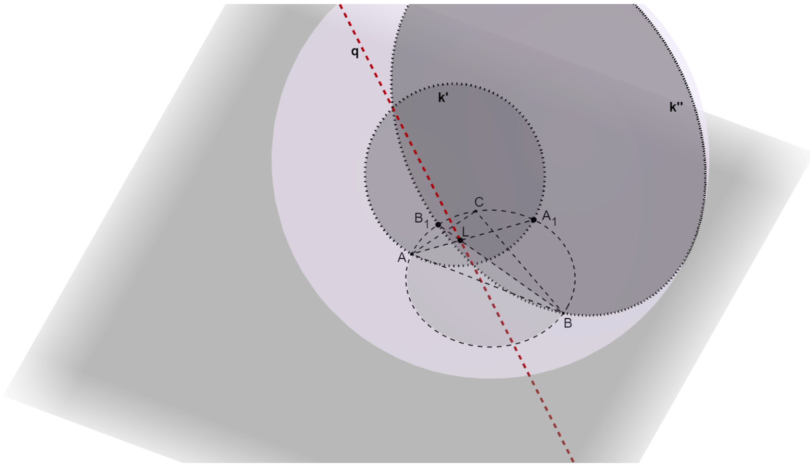

Figure 5

point B_1. But $B_1 \in k'' \in \Sigma_B$, and B_1 lies on the plane through $\triangle ABC$. Therefore, B_1 lies on the Apollonian circle corresponding to the vertex B. Therefore, from planimetry, we know that BB_1 is a symmedian in $\triangle ABC$.

Therefore, $AA_1 \cap BB_1 = L$.

It remains to be considered that as $L \in \alpha$, and $L \in \beta$, then $L \in q$ (the intersection of the two planes). □

Note. Another approach to proving the geometric fact is via properties of polar reciprocation. From planimetry, (Andreescu et al. 2016, p. 279), we know that L is the pole of the line s, and from a), we know that the lines s and q are polar, meaning that $L \in q$.

c) Let $k' \cap k'' = \{K, T\}$ (fig. 6). Then we know that $K \in (\alpha \cap \beta)$, and $T \in (\alpha \cap \beta)$, from where it follows that $KT \equiv q$. We note that $q \in \gamma$, that is, $KT \in \gamma$. But $\{K, T\} \in \Sigma \Rightarrow \{K, T\} \in (\gamma \cap \Sigma) \equiv k'''$. Also $\{K, T\} \in k', k'', k''' \in \Sigma_A, \Sigma_B, \Sigma_C \Rightarrow \{K, T\} \in \Sigma_A \cap \Sigma_B \cap \Sigma_C$, which by Theorem 2.2., is a circle, which we will denote by Ω, and the plane passing through it – by μ. Therefore, $KT \in \mu$. $KT \equiv q \Rightarrow q \in \mu$.

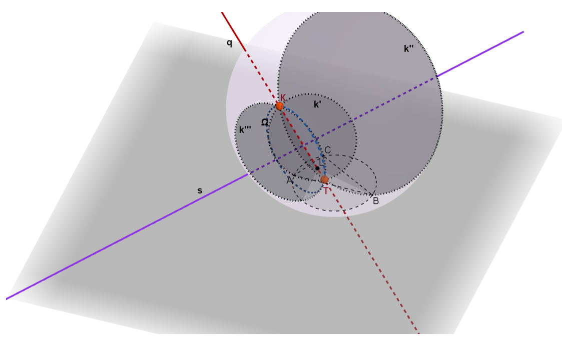

Figure 6

But μ is the radical plane of the three Apollonian spheres. Therefore, the line through their centers (s) is perpendicular to μ. Thus, it is perpendicular to every line in it, and q is a line in it. Therefore, $s \perp q$. □

Note. The statement could also be proved via the property of the polar reciprocation in 3D space that two polar lines with respect to a given sphere are perpendicular, as we know from a) that the lines s and q are polar.

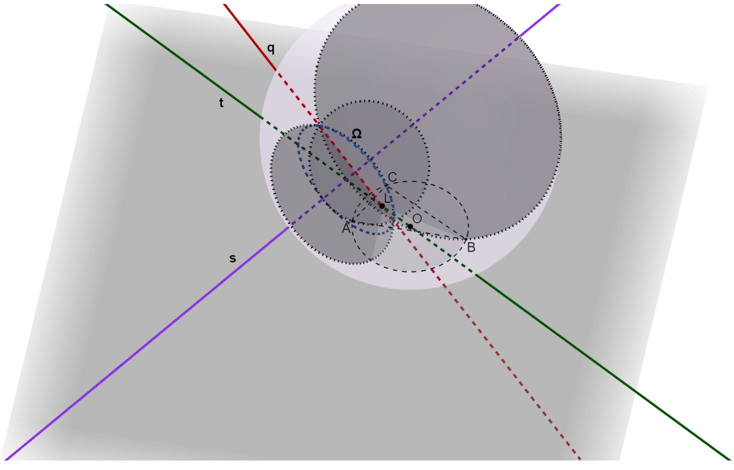

Figure 7

d) Let us denote the orthogonal projection of the line q onto the plane passing through $\triangle ABC$, δ_{ABC}, by t (fig. 7). From planimetry, we know that the Lemoine axis is perpendicular to the Brocard axis: $s \perp OL$. We notice that $s \perp \mu$, and $s \in \delta_{ABC}$, from where it follows that $\delta_{ABC} \perp \mu$. From the orthogonality of the two planes, we deduce that $t \in \mu$ since t is their intersection. From $s \perp \mu$, it follows that $s \perp t$. As $L \in \delta_{ABC}$ and $L \in q$, it is clear that $L \in t$. But there is a single line in δ_{ABC}, perpendicular to s and passing through L, and OL is such. Therefore, $OL \equiv t \Leftrightarrow O \in t$. □

Note. The statement that $s \perp t$ could also be derived by the three-perpendiculars theorem since $s \perp q$.

3. Two Notable Properties of Stereographic Projection

Alongside the analogies from the plane in 3D space, the Apollonian sphere finds additional applications in proving new notable properties of stereographic projection. Stereographic projection is a projection of a sphere upon a plane such that the center of the projection lies on the sphere, and the plane is perpendicular to the diameter of the sphere through the given point. For stereographic projection, it is well known that the center of projection, the centroid of the projection triangle, and the centroid of the projected triangle are not collinear, as well as for their orthocenters and circumcenters, but it appears that the centers of the Apollonian circles and the Lemoine point fulfill this property.

Theorem 3.1. Let under stereographic projection with a projection point O, a reference sphere Σ, and a projection plane μ, $\triangle ABC$ map to $\triangle A'B'C'$. Then point O, the center of an Apollonian circle for $\triangle ABC$, and the center of the corresponding Apollonian circle for $\triangle A'B'C'$ are collinear.

Proof. Firstly, we will introduce the following Lemma (Prasolov 2010, p. 313):

Lemma: Let under inversion with a point of inversion O, the circle k map to the circle k'. Let the sphere Σ pass through point O and the circle k. Then point O, the pole of the plane passing through the circle k with respect to the sphere Σ, and the center of k' are collinear.

Back to our main problem: as by condition, Σ is a reference sphere, and μ – a projection plane, then, by definition of stereographic projection, it follows that the plane μ is perpendicular to the diameter in Σ through O. Therefore, there exists a single sphere $i(O, r)$ with a center point O such that the plane μ is the radical plane of the spheres Σ and i. Therefore, under inversion p with an inversion sphere $i(O, r)$, $\Sigma \xrightarrow{p} \mu$ (and so $A \xrightarrow{p} A'$, $B \xrightarrow{p} B'$, and $C \xrightarrow{p} C'$).

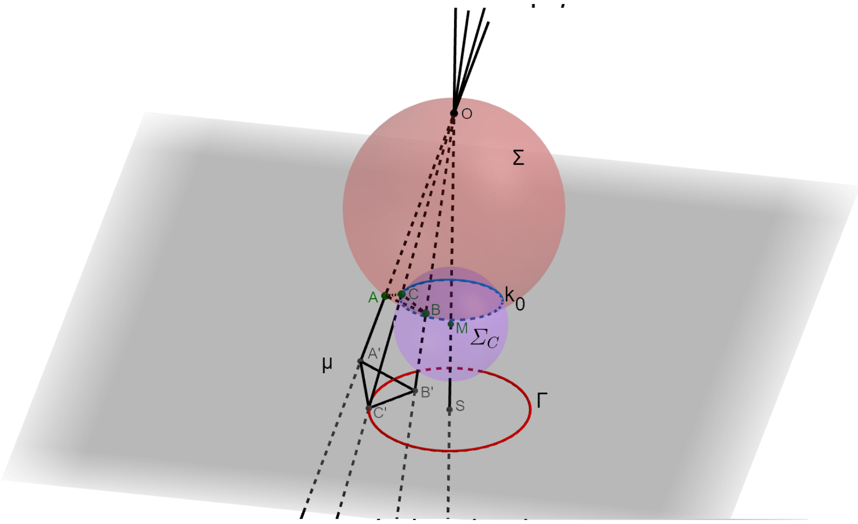

Figure 8

Let point M be the center of an Apollonian circle for $\triangle ABC$, and let $OM \cap \mu = S$ (see Fig. 8). We will prove that point S is a center of an Apollonian circle for $\triangle A'B'C'$.

Let us denote an Apollonian circle for $\triangle ABC$ by k_C, and the corresponding Apollonian circle for $\triangle A'B'C'$ by Γ. Let us also construct the Apollonian sphere Σ_C for $\triangle ABC$. As we have taken k_C with a ratio $\frac{CA}{CB}$, then, Σ_C and Γ will refer to the ratios $\frac{CA}{CB}$ and $\frac{C'A'}{C'B'}$, respectively. We note that point M is also the center of Σ_C (the Apollonian circles are great circles in the Apollonian spheres). Let $\Sigma \cap \Sigma_C = k_0$.

Firstly, we will prove that $k_0 \xrightarrow{p} \Gamma \quad (*)$.

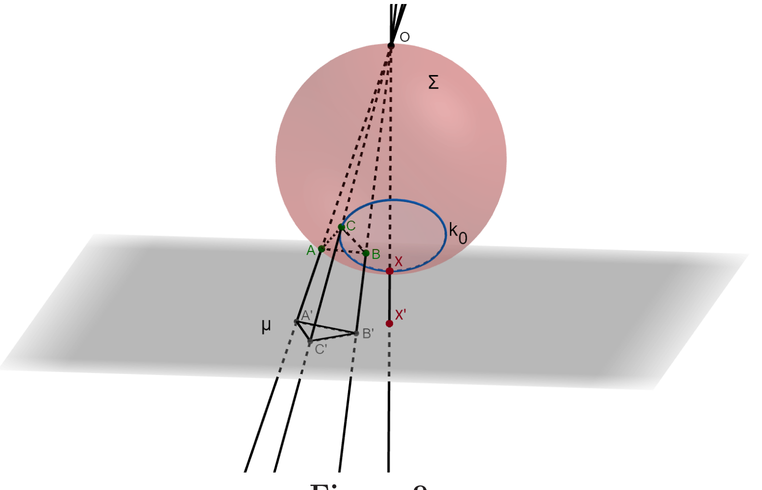

Figure 9

Let us take an arbitrary point X from k_0, and let $X \xrightarrow{p} X'$ (see fig. 9). It suffices to show that $X' \in \Gamma$.

$X \in k_0 \in \Sigma \Rightarrow X' \in \mu \equiv \Sigma' \Rightarrow OX \cap \mu = X'$

$X \in k_0 \in \Sigma_C \xRightarrow{2.1.}$

$$\frac{XA}{XB} = \frac{CA}{CB} \quad (1)$$

Without loss of generality, let $r = 1$.

The lines $A'O$ and $X'O$ form a plane. We have that $OA \cdot OA' = r^2 = 1 = OX \cdot OX' \Rightarrow \dfrac{OA}{OX} = \dfrac{OX'}{OA'}$ and $\angle AOX = \angle X'OA' \Rightarrow \triangle OAX \sim \triangle OX'A' \Rightarrow$

$$\frac{AX}{X'A'} = \frac{OA}{OX'} \Leftrightarrow X'A' = \frac{AX \cdot OX'}{OA} \quad (2).$$

The lines $B'O$ and $X'O$ form a plane. We have that $OB \cdot OB' = r^2 = 1 = OX \cdot OX' \Rightarrow \dfrac{OB}{OX} = \dfrac{OX'}{OB'}$ and $\angle BOX = \angle X'OB' \Rightarrow \triangle OBX \sim \triangle OX'B' \Rightarrow$

$$\frac{BX}{X'B'} = \frac{OB}{OX'} \Leftrightarrow X'B' = \frac{BX \cdot OX'}{OB} \quad (3).$$

By (2) and (3) $\Rightarrow$

$$\frac{X'A'}{X'B'} = \frac{AX}{BX} \cdot \frac{OB}{OA} \stackrel{(1)}{=} \frac{CA}{CB} \cdot \frac{OB}{OA}.$$

But point $C' \in \Gamma$, and by applying the metric property of inversion, (Johnson 1960, p. 48), it follows that:

$$\frac{C'A'}{C'B'} = \frac{\frac{CA}{OC \cdot OA}}{\frac{CB}{OC \cdot OB}} = \frac{CA}{CB} \cdot \frac{OB}{OA}.$$

$$\Rightarrow \frac{X'A'}{X'B'} = \frac{C'A'}{C'B'}$$

It remains to be considered that $X' \in \mu$. Therefore, from planimetry, we know that point X' lies on the Apollonian circle in the plane μ with a ratio $\frac{C'A'}{C'B'} \equiv \Gamma$. $\Rightarrow X' \in \Gamma$ that we wanted to prove.

Secondly, we will prove that point M is the pole of the plane passing through the circle k_0 with respect to Σ. $\quad (**)$

By Theorem 2.3., we know that $\Sigma_C \perp \Sigma$, and by construction, $k_0 = \Sigma_C \cap \Sigma$. This, combined with the fact that point M is the center of Σ_C, proves the statement.

By $(*)$ and $(**)$ $\xRightarrow{\textbf{Lemma}}$ The points O, M, and the center of Γ are collinear. Therefore, the line OM intersects the plane through the circle Γ at its center. But Γ and $\triangle A'B'C'$ lie on a plane (μ). Consequently, OM intersects μ at the center of the Apollonian circle for $\triangle A'B'C'$, and by construction, $OM \cap \mu = S$ $\Rightarrow$ point S is the center of Γ that we wanted to show. $\square$

Note. Point M does not project onto point S.

Let us prove the following notable fact as well, which resembles and is partly based on the previous one.

Theorem 3.2. Let under stereographic projection with a projection point O, a reference sphere Σ, and a projection plane μ, $\triangle ABC$ map to $\triangle A'B'C'$. Then point O, the Lemoine point for $\triangle ABC$, and the Lemoine point for $\triangle A'B'C'$ are collinear.

Proof. Firstly, we will introduce and prove the following Lemma.

Lemma: Let p be an inversion with an inversion circle $i(O, R)$, and let k be a circle not passing through O, on which lie the points A, B, C, D, E, and F such that $AD \cap BE \cap CF = Q$. Let under p, the points A, B, C, D, E, and F map to the points A', B', C', D', E', and F'. Then the lines $A'D'$, $B'E'$,

$C'F'$, and OQ concur at one point.

Proof. Let us denote the inverse point of Q by Q' (see fig. 10).

By condition, the circle k does not pass through the center of inversion, point O. Therefore, it will be mapped again to a circle. And so the points lying on k will be sent to points lying on k' – the image of k. Thus, the points A', B', C', D', E', and F' lie on a circle, k'.

We know that under inversion p, the line AD will be sent to a circle, passing through O. Therefore, the points A', Q', D', and O lie on a circle, which we will denote by k_1. Analogously, the points B', Q', E', and O lie on the circle k_2, and the points C', Q', F', and O lie on the circle k_3.

Let us consider the circles k', k_1, and k_2. As the three are intersecting at three pairs of points, then their common chords are intersecting at one point (Prasolov 2010a, p. 269). Therefore, $Q'O \cap B'E' \cap A'D' = S$.

Analogously, we consider the circles k', k_2, and k_3, which also intersect at three pairs of points, meaning that their common chords are intersecting at one point. Thus, $Q'O \cap C'F' \cap B'E' = S'$.

But $S' = Q'O \cap B'E' = S \Rightarrow S' \equiv S \Rightarrow$ As $OQ \equiv OQ'$, $OQ \cap A'D' \cap B'E' \cap C'F' = S$, which we wanted to show. □

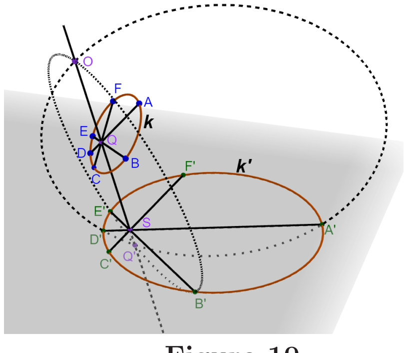

Figure 10

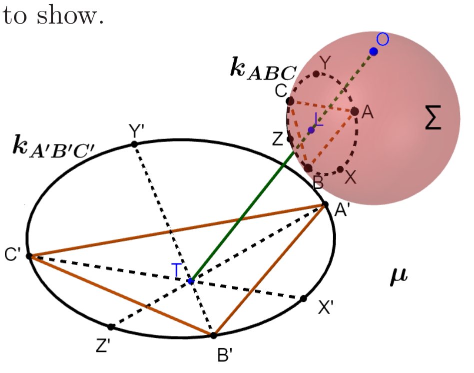

Figure 11

Back to our main proof: as by condition, Σ is a reference sphere, and μ - a projection plane, then, by definition of stereographic projection, it follows that the plane μ is perpendicular to the diameter in Σ through O. Therefore, there exists a single sphere $i(O, r)$ with a center point O such that the plane μ is the radical plane of the spheres Σ and i. Therefore, under inversion p with an inversion sphere $i(O, r)$, $\Sigma \xrightarrow{p} \mu$ (and so $A \xrightarrow{p} A'$, $B \xrightarrow{p} B'$, and $C \xrightarrow{p} C'$).

Let point L be the Lemoine point for $\triangle ABC$, and let $OL \cap \mu = T$ (see fig. 11). We will prove that point T is the Lemoine point for $\triangle A'B'C'$.

Let us define k_C, Γ, Σ_C, and k_0 as in Theorem 3.1. Let us denote the circumcircles of $\triangle ABC$ and $\triangle A'B'C'$ by k_{ABC} and $k_{A'B'C'}$, respectively. Since a circle not passing through the center of inversion is sent to a circle, and the points A, B, and C are sent to the points A', B', and C', then, $k_{ABC} \xrightarrow{\text{p}} k_{A'B'C'}$.

Let $k_{ABC} \cap k_C = \{C, X\}$. Point $X \in k_C \Rightarrow X \in \Sigma_C$. But point $X \in k_{ABC} \Rightarrow X \in \Sigma$. Therefore, point $X \in (\Sigma_C \cap \Sigma) = k_0$. Hence, as $k_{ABC} \xrightarrow{\text{p}} k_{A'B'C'}$, and $X \in k_{ABC}$, then X' (the inverse point of X) $\in k_{A'B'C'}$. However, since by Theorem 3.1., $k_0 \xrightarrow{\text{p}} \Gamma$, and $X \in k_0$, then $X' \in \Gamma$. Consequently, $X' \in (k_{A'B'C'} \cap \Gamma)$, and so $k_{A'B'C'} \cap \Gamma = \{C', X'\}$.

It remains to be considered that from planimetry, we know that CX and $C'X'$ are symmedians in $\triangle ABC$ and $\triangle A'B'C'$, respectively. Analogously, we define the points Y and Z, from which it follows that BY and AZ are symmedians in $\triangle ABC$, and $B'Y'$ and $A'Z'$ – in $\triangle A'B'C'$. Furthermore, this means that $CX \cap BY \cap AZ = L$, and $C'X' \cap B'Y' \cap A'Z' = T'$, which are Lemoine points for $\triangle ABC$ и $\triangle A'B'C'$, respectively, and we also know that the points A, B, C, X, Y, Z lying on a circle are mapped to the points A', B', C', X', Y', Z', and the lines formed by the opposite points concur at one point (L and T', respectively) $\xRightarrow{\textbf{Lemma}} T' \in OL$. But $T' \in \mu$, and the line OL intersects the plane μ at a single point, which by construction is point T. Therefore, $T' \equiv T$, which we wanted to prove. □

Note. Point L does not project onto point T.

4. Applications in Olympiad Geometry

The Apollonian sphere and its key results seen in Section 3 connect with Olympiad geometry. Firstly, we discover a configuration of Olympiad-level complexity by finding the planimetric equivalent of the aforementioned notable facts around stereographic projection. Secondly, we provide an original solution to another Olympiad problem using the Apollonian sphere.

We could reformulate the statements in Theorems 3.1. and 3.2. by the following way: "Let us take in 3D space the intersecting circles k_1, k_2, and k_3 such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Then point O, the center of an Apollonian circle/Lemoine point for $\triangle ABC$, and the center of the corresponding Apollonian circle/Lemoine point for $\triangle DEF$ are collinear." Point O lies on the radical line of the three circles, and so it will have the same power with respect to them, S. Therefore, we can take an inversion with an inversion sphere with a center point O and a radius $\sqrt{S}$, at which $\triangle ABC$ will map to $\triangle DEF$. It is clear that there exists a stereographic projection with a projection point O, a reference sphere passing through the points A, B, C, and O, and a projection plane passing through the points D, E, and F. As a result, the restatement is valid.

As a consequence, for the particular case when the three circles lie on a plane, we will get the following problem:

Problem 4.1. Let us take in the plane the intersecting circles k_1, k_2, and k_3 such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the Lemoine point for $\triangle ABC$, and the Lemoine point for $\triangle DEF$ are collinear (fig. 12).

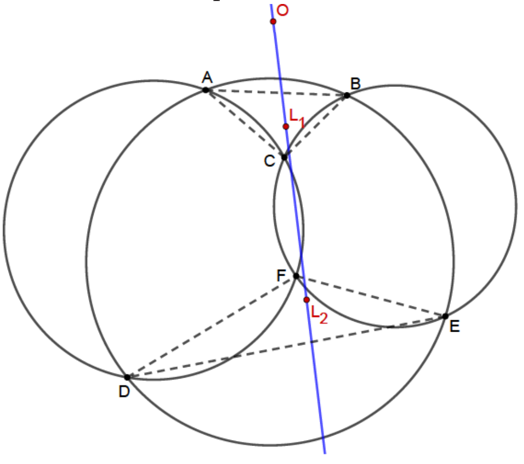

Figure 12

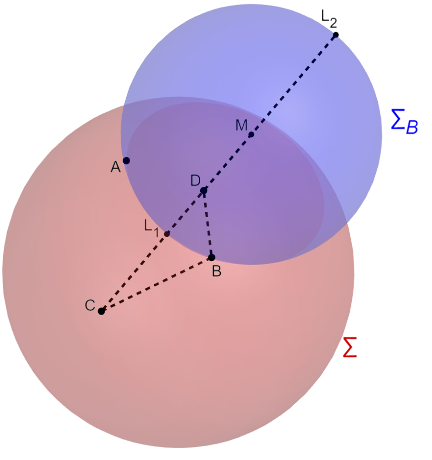

Figure 13

Using Apollonian sphere, we give an authentic solution to the following Olympiad problem.

Problem 4.2. ***(30-th Bulgarian Mathematical Olympiad 1981, Fourth Round, Second Day, Sixth Problem)*** Planes $\alpha, \beta, \gamma, \delta$ are tangent to the circumsphere of a tetrahedron $ABCD$ at points A, B, C, D, respectively. Line p is the intersection of α and β, and line q is the intersection of γ and δ. Prove that if lines p and CD meet, then lines q and AB lie on a plane.

Proof. Let Σ denote the circumsphere of $ABCD$. Let us take $\triangle BCD$ with interior and exterior angle bisectors of $\angle CBD$ BL_1 and BL_2, respectively (see fig. 13). Let point M be the midpoint of L_1L_2. It is clear that we have $\Sigma_B(M, r = MB = ML_1 = ML_2)$. Since by condition, the points $B, C, D \in \Sigma$ and by construction, Σ_B is an Apollonian sphere for $\triangle BCD$, then by Theorem 2.3., it follows that $\Sigma_B \perp \Sigma$. Therefore, $M \in \beta$. But point $M \in CD$, and $CD \cap \beta$ at a single point $\Rightarrow$ as by condition, $CD \cap \beta \cap \alpha$, then $M \in \alpha \Rightarrow A \in \Sigma_B \overset{2.1.}{\Longrightarrow} \dfrac{AC}{AD} = \dfrac{BC}{BD} \Leftrightarrow \dfrac{AC}{BC} = \dfrac{AD}{BD} = \zeta$.

If $\zeta = 1$, $\gamma \parallel AB$, and $\delta \parallel AB$, meaning that $q \parallel AB$, and so q and AB lie on a plane.

If $\zeta \neq 1$, by Theorem 2.1., point C lies on the Apollonian sphere for $\triangle BAD$ with a ratio $\frac{DA}{DB}$, Σ_D. Let us take $\triangle BAD$ with interior and exterior angle bisectors of $\angle BDA$ DJ_1 and DJ_2, respectively. Let point N be the midpoint of J_1J_2. It is clear that $\Sigma_D(N, r = ND = NJ_1 = NJ_2)$. By Theorem 2.3., $\Sigma_D \perp \Sigma \Rightarrow N \in \delta$, and $N \in \gamma \Rightarrow N \in (\delta \cap \gamma) \equiv q \Rightarrow N \in q$. But by construction, $N \in AB \Rightarrow AB \cap q = N \Rightarrow AB$ and q lie on a plane. □

Note: This tetrahedron is the 3D equivalent of the harmonic quadrilateral, (Boev 2010, pp. 76 – 87), due to its distinctive connection between tangency and metric relation, and so we can call it a *harmonic tetrahedron*.

5. Conclusion

As we see, the Apollonian sphere is the equivalent of the Apollonian circle in 3D space due to the similarity of their definitions and some of their properties, namely orthogonality and coaxiality. Additionally, we find a stereometric parallel of the connection between the Apollonian circle and the notable points of the triangle. We examine the intersecting planes of the Apollonian spheres with a sphere through the given triangle, which helps us establish the connection between the Apollonian sphere, the Lemoine point, and the circumcenter of the triangle. The planes intersect at a line, q, that we prove is polar and thus, perpendicular to the central axis of the spheres, and its orthogonal projection coincides with the Brocard axis.

The Apollonian sphere holds significant applicability in solid geometry. Using it, we prove the two newly discovered notable properties of stereographic projection illustrated in Section 3. We prove collinearity between the two centers of corresponding Apollonian circles or two Lemoine points with the projection center. We note that under stereographic projection the two Lemoine points and the two centers of corresponding Apollonian circles are not image and preimage. That is why we call their properties notable. In the proof of the given stereometric construction, we use the Apollonian sphere both to prove that its common circle with the reference sphere projects onto the second Apollonian circle (in Theorem 3.1 and 3.2) and to prove that the center of the first Apollonian circle is the pole of the plane through the common circle between the Apollonian sphere and the reference sphere with respect to the reference sphere (in Theorem 3.1).

We establish the connection of these new geometric constructions with Olympiad geometry, as seen in Section 4. First, we reformulate the two theorems from Section 3 such that we include the particular case when the configuration is planimetric, which we separate as an Olympiad problem. Second, we apply the Apollonian sphere to another Olympiad problem, which gives us an authentic proof of it.

Consequently, in this paper, we managed to: define the concept Apollonian sphere (Theorem 2.1), discover and prove its properties (Theorems 2.2, 2.3, and 2.4), find the geometric configurations around the two notable properties of stereographic projection, (Theorems 3.1 and 3.2), prove them via an Apollonian sphere, formulate an Olympiad problem derived from those properties (Problem 4.1), and an original solution to another Olympiad problem via an Apollonian sphere (Problem 4.2).

In conclusion, we define a new concept, and after revealing its properties, we show its significance in geometry through the revelation of new configurations. We also formulate an Olympiad problem and solve another one using the newly discovered concepts. The aforementioned configurations would also be crucial both for the development of Olympiad geometry and for stimulating others to develop further on the topic.

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NOTES

1. All statements in this paper refer to the Euclidean three-dimensional space unless otherwise stated.

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