

AN INTERESTING ALGEBRAIC INEQUALITY AND SOME APPLICATIONS

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Abstract. The paper consider the following algebraic inequality

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}, \quad (1)$$

where x, y, z are positive numbers and $a, b, c \in \mathbb{R}$. Some generalizations are proposed and examples of application are discussed.

Keywords: algebraic inequality, inequality of Cauchy-Bunyakovski-Schwarz, generalization, application

The general form of (1) is the inequality

$$\sum_{i=1}^n \frac{a_i^2}{x_i} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{x_1 + x_2 + \dots + x_n},$$

where $x_i > 0$ and $a_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. In fact, this is the well-known Cauchy-Bunyakovski-Schwarz inequality in the so called **Engel¹ form**. It became very popular among the American students who attended the training course of the USA IMO team. This happend after a lecture delivered by the past leader of the team Titu Andreescu at the Mathematical Olympiad Summer Program (MOSP) held at Georgetown University in June, 2001.

It is easy to prove the inequality in the simplest case $n = 2$, i.e. to prove that:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}, \quad (2)$$

where $x, y > 0$ and $a, b \in \mathbb{R}$. Multiplying both sides of (2) by $xy(x+y) > 0$, we get the equivalent inequality:

¹ Engel, Arthur – German mathematician, long time one of main trainer of the German IMO team; the author of well-known book „Problem-Solving Strategies“, Springer, 1998.

$$(ay - bx)^2 \geq 0,$$

which is true, obviously. We have an equality in (2) if and only if $\frac{a}{x} = \frac{b}{y}$. Further, using (2) it is easy to prove the general form of (1) by induction, what is demonstrated in (Arslanagić, 2008), (Grozdev, 2005) and (Grozdev, 2007):

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \dots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{x_1 + x_2 + \dots + x_n}, \quad (3)$$

where $x_i > 0$ and $a_i \in \mathbb{R}$, $i = 1, 2, \dots, n$. The equality holds true in (1) and in (3) if and only if:

$$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}, \text{ and } \frac{a_1}{x_1} = \frac{a_2}{x_2} = \dots = \frac{a_n}{x_n}, \text{ respectively.}$$

Remark 1. If in (1) we make the substitution $a = a_1 b_1$, $b = a_2 b_2$, $c = a_3 b_3$, $x = b_1^2$, $y = b_2^2$, $z = b_3^2$, where $x, y, z > 0$ and $a, b, c \in \mathbb{R}$, we get the well-known Cauchy-Bunyakovski-Schwarz inequality for $n = 3$:

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

Analogously, from (3) we get the generalized Cauchy-Bunyakovski-Schwarz inequality:

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2).$$

It is given in (Arslanagić & Baši, 2014) a proof of the inequality:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^2}{3(x+y+z)},$$

where $a, b, c, x, y, z > 0$. A generalization is proposed in (Milošević, 2014):

$$\frac{a^n}{x} + \frac{b^n}{y} + \frac{c^n}{z} \geq \frac{(a+b+c)^n}{3^{n-2}(x+y+z)},$$

where $a, b, c, x, y, z > 0$ and $n \geq 2, n \in \mathbb{N}$.

Now, we will give some interesting examples of application.

Example 1. If $x, y, z > 0$ and $2xyz = I$, then the following inequality holds true:

$$\frac{xy^2}{x^3 + I} + \frac{yz^2}{y^3 + I} + \frac{zx^2}{z^3 + I} \geq I. \quad (4)$$

Proof: Because $2xyz = I$, we get by (1):

$$\begin{aligned} \frac{xy^2}{x^3 + I} + \frac{yz^2}{y^3 + I} + \frac{zx^2}{z^3 + I} &= \frac{y^2}{x^2 + \frac{I}{x}} + \frac{z^2}{y^2 + \frac{I}{y}} + \frac{x^2}{z^2 + \frac{I}{z}} \geq \\ &\geq \frac{(y + z + x)^2}{x^2 + y^2 + z^2 + \frac{I}{x} + \frac{I}{y} + \frac{I}{z}} = \frac{(x + y + z)^2}{x^2 + y^2 + z^2 + \frac{yz + xz + xy}{xyz}} = \\ &= \frac{(x + y + z)^2}{x^2 + y^2 + z^2 + 2xy + 2yz + 2zx} = \frac{(x + y + z)^2}{(x + y + z)^2} = I. \end{aligned}$$

Thus, the inequality is proved. The equality holds in (4) if and only if $x = y = z \Rightarrow 2x^3 = I \Rightarrow x = y = z = \sqrt[3]{\frac{I}{2}}$.

Example 2. If $x, y, z > 0$ and $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = I$, then following inequality holds:

$$\frac{x^2}{x + y} + \frac{y^2}{y + z} + \frac{z^2}{z + x} \geq \frac{I}{2}. \quad (5)$$

Proof: Applying (1), we get:

$$\begin{aligned} \frac{x^2}{x + y} + \frac{y^2}{y + z} + \frac{z^2}{z + x} &\geq \frac{(x + y + z)^2}{2(x + y + z)}, \text{ i.e.} \\ \frac{x^2}{x + y} + \frac{y^2}{y + z} + \frac{z^2}{z + x} &\geq \frac{x + y + z}{2}. \end{aligned} \quad (6)$$

By the AM-GM inequality for two positive numbers we have:

$$\frac{x+y}{2} \geq \sqrt{xy}; \quad \frac{y+z}{2} \geq \sqrt{yz}; \quad \frac{z+x}{2} \geq \sqrt{zx}.$$

It is enough to add the left-hand and right-hand sides:

$$2(x+y+z) \geq 2(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}), \text{ i.e.}$$

$$x+y+z \geq I, \tag{7}$$

because $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = I$.

The equality holds true in (5) if and only if $x = y = z = \frac{I}{3}$.

Example 3. If $a, b, c > 0$, then the following inequality holds true:

$$\frac{a}{3b^2 + 2bc + 3c^2} + \frac{b}{3c^2 + 2ca + 3a^2} + \frac{c}{3a^2 + 2ab + 3b^2} \geq \frac{I}{4} \left(\frac{I}{a+b} + \frac{I}{b+c} + \frac{I}{c+a} \right). \tag{8}$$

Proof. Using the inequality (1), we have:

$$\begin{aligned} & \frac{a}{3b^2 + 2bc + 3c^2} + \frac{b}{3c^2 + 2ca + a^2} + \frac{c}{3a^2 + 2ab + 3b^2} = \\ &= \frac{a^2}{3a(b^2 + c^2) + 2abc} + \frac{b^2}{3b(c^2 + a^2) + 2abc} + \frac{c^2}{3c(a^2 + b^2) + 2abc} \geq \\ & \geq \frac{(a+b+c)^2}{3(ab^2 + a^2b + a^2c + ac^2 + b^2c + bc^2 + 2abc)} = \\ &= \frac{(a+b+c)^2}{3(a+b)(b+c)(c+a)} = \frac{I}{4} \cdot \frac{[(a+b) + (b+c) + (c+a)]^2}{3(a+b)(b+c)(c+a)} \\ & \geq \frac{I}{4} \cdot \frac{3[(a+b)(b+c) + (b+c)(c+a) + (c+a)(a+b)]}{3(a+b)(b+c)(c+a)} = \\ &= \frac{I}{4} \left(\frac{I}{a+b} + \frac{I}{b+c} + \frac{I}{c+a} \right). \end{aligned}$$

Thus, the inequality under consideration is proved. The equality holds in (8) if and only if $a = b = c$.

Remark 2. In the proof of the inequality (8) we have used the well-known inequality:

$$(x + y + z)^2 \geq 3(xy + yz + zx); \quad (x, y, z \in \mathbb{R})$$

$$\Leftrightarrow x^2 + y^2 + z^2 \geq xy + yz + zx$$

$$\Leftrightarrow \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \geq 0.$$

Example 4. If $a, b, c > 0$ and $a^4 + b^4 + c^4 = ab + bc + ca$, then the following inequality holds true:

$$\frac{a^2b}{b^3+c} + \frac{b^2c}{c^3+a} + \frac{c^2a}{a^3+b} \geq \frac{1}{2}(ab + bc + ca). \quad (9)$$

Proof: Using the inequality (1) and the given condition, we get $a^4 + b^4 + c^4 = ab + bc + ca$:

$$\begin{aligned} \frac{a^2b}{b^3+c} + \frac{b^2c}{c^3+a} + \frac{c^2a}{a^3+b} &= \frac{a^2b^2}{b^4+bc} + \frac{b^2c^2}{c^4+ac} + \frac{c^2a^2}{a^4+ab} \geq \\ &\geq \frac{(ab + bc + ca)^2}{a^4 + b^4 + c^4 + ab + bc + ca} = \frac{(ab + bc + ca)^2}{2(ab + bc + ca)} = \\ &= \frac{1}{2}(ab + bc + ca), \text{ q.e.d.} \end{aligned}$$

The equality holds in (9) if and only if $a = b = c = 1$.

Example 5. If $x, y, z > 0$, then the following inequality holds true:

$$\frac{(2x + y + z)^2}{x + 2y + z} + \frac{(x + 2y + z)^2}{y + x + 2z} + \frac{(x + y + 2z)^2}{2x + y + z} \geq 4(x + y + z). \quad (10)$$

Proof: It follows from (1) that:

$$\begin{aligned}
 & \frac{(2x+y+z)^2}{x+2y+z} + \frac{(x+2y+z)^2}{y+x+2z} + \frac{(x+y+2z)^2}{2x+y+z} \geq \\
 & \geq \frac{(2x+y+z+x+2y+z+x+y+2z)^2}{x+2y+z+y+x+2z+2x+y+z} \\
 & = \frac{(4x+4y+4z)^2}{4x+4y+4z} = 4(x+y+z), \text{ q.e.d.}
 \end{aligned}$$

The equality holds in (10) if and only if $x = y = z$.

Remark 3. In (Bencze & Arslanagić, 2008) the inequality (10) is proved using the following inequality:

$$\frac{(2x+y+z)^2}{x+2y+z} \geq 3x+z \quad (11)$$

$$(\Leftrightarrow 4x^2 + y^2 + z^2 + 4xy + 4xz + 2yz \geq 3x^2 + 6xy + 3xz + xz + 2yz + z^2)$$

$$(\Leftrightarrow x^2 + y^2 - 2xy \geq 0)$$

$$(\Leftrightarrow (x-y)^2 \geq 0).$$

We have two analogous inequalities and after adding them, we get (10). But, the question is: „How does the inequality (11) appear?“

Example 6. If $x, y, z > 0$, then the following inequality holds true:

$$\frac{x^2}{y(x^2+xy+y^2)} + \frac{y^2}{z(y^2+yz+z^2)} + \frac{z^2}{x(z^2+zx+x^2)} \geq \frac{3}{x+y+z}. \quad (12)$$

Proof: Using the inequality (1) we get:

$$\begin{aligned}
 & \frac{x^2}{y(x^2 + xy + y^2)} + \frac{y^2}{z(y^2 + yz + z^2)} + \frac{z^2}{x(z^2 + zx + x^2)} \geq \\
 & \geq \frac{(x + y + z)^2}{yx^2 + xy^2 + y^3 + zy^2 + yz^2 + z^3 + xz^2 + zx^2 + x^3} = \\
 & = \frac{(x + y + z)^2}{x^2(x + y + z) + y^2(x + y + z) + z^2(x + y + z)} = \\
 & = \frac{(x + y + z)^2}{(x + y + z)(x^2 + y^2 + z^2)} = \frac{x + y + z}{x^2 + y^2 + z^2}.
 \end{aligned}$$

Now, we have to prove the inequality:

$$\begin{aligned}
 & \frac{x + y + z}{x^2 + y^2 + z^2} \geq \frac{3}{x + y + z}, \text{ i.e.} \\
 & (x + y + z)^2 \geq 3(x^2 + y^2 + z^2) \\
 & \Leftrightarrow xy + yz + zx \geq x^2 + y^2 + z^2,
 \end{aligned}$$

which is not exact because the following inequality holds true:

$$x^2 + y^2 + z^2 \geq xy + yz + zx; \quad (x, y, z \in \mathbb{R}).$$

This inequality is not enough powerful to prove (12). The important question is how to prove the inequality (12)? Is (12) true or not-true? The answer is affirmative, i.e. this inequality is true. For the proof we will use the inequality:

$$\begin{aligned}
 & \frac{x^2}{y(x^2 + xy + y^2)} \geq \frac{2x - y}{3xy}; \quad (x, y > 0) \tag{13} \\
 & \Leftrightarrow 3x^3 \geq (2x - y)(x^2 + xy + y^2) \\
 & \Leftrightarrow x^3 \geq x^2y + xy^2 - y^3 \\
 & \Leftrightarrow x^3 + y^3 - xy(x + y) \geq 0
 \end{aligned}$$

$$\Leftrightarrow (x+y)(x-y)^2 \geq 0.$$

Using now the inequality (13) and its analogues, we get:

$$\begin{aligned} & \frac{x^2}{y(x^2+xy+y^2)} + \frac{y^2}{z(y^2+yz+z^2)} + \frac{z^2}{x(z^2+zx+x^2)} \geq \\ & \geq \frac{2x-y}{3xy} + \frac{2y-z}{3yz} + \frac{2z-x}{3xz} = \\ & = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{z} + \frac{1}{x} \right) - \frac{1}{3} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \\ & = \frac{1}{3} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right). \end{aligned} \tag{14}$$

Apply now the AM-HM inequality:

$$\begin{aligned} \frac{x+y+z}{3} & \geq \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}, \text{ i.e.} \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} & \geq \frac{9}{x+y+z}. \end{aligned} \tag{15}$$

The equality holds in (12) if and only if $x = y = z$.

We recommend to the readers of the present paper to try proving the next inequalities using (1) or (3):

$$1. \frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}; \quad (a, b, c > 0).$$

(Tournament of the Towns, 1998)

$$2. \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}; (a, b, c > 0 \text{ i } abc = 1). \quad (\text{IMO 1995})$$

$$3. \frac{x^2+y^2+z^2}{x^5+y^2+z^2} + \frac{x^2+y^2+z^2}{x^2+y^5+z^2} + \frac{x^2+y^2+z^2}{x^2+y^2+z^5} \leq 3; (x, y, z > 0 \text{ and } xyz \geq 1). \quad (\text{IMO 2005})$$

$$4. \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}; (a, b, c, d > 0 \text{ i } a+b+c+d = 1). \quad (\text{Ireland, 1999})$$

$$5. \frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}; (a, b, c > 0 \text{ and } a^2+b^2+c^2 = 3abc). \quad (\text{India})$$

$$6. \frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1; (a, b, c > 0). \quad (\text{Czech-Slovak Competition, 1990})$$

$$7. \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{9}{10}; (a, b, c > 0 \text{ and } a+b+c = 1). \quad (\text{India})$$

$$8. \frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}; (a, b, c, d > 0 \text{ and } ab+bc+cd+da = 1). \quad (\text{IMO 1990, Shortlist})$$

$$9. \frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}; (a, b, c, d > 0). \quad (\text{Titu Andreescu, IMO 1993, Shortlist})$$

$$10. \frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \frac{a_3}{a_4+a_5} + \frac{a_4}{a_5+a_1} + \frac{a_5}{a_1+a_2} \geq \frac{5}{2}; (a_1, a_2, a_3, a_4, a_5 > 0).$$

$$11. \frac{(b+c-a)^2}{a^2+(b+c)^2} + \frac{(c+a-b)^2}{b^2+(c+a)^2} + \frac{(a+b-c)^2}{c^2+(a+b)^2} \geq \frac{3}{5}; (a, b, c > 0).$$

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