Educational Research
Научно–методически статии

AN INEQUALITY FOR A RIGHT TRIANGLE AND ITS GENERALIZATION

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Abstract. The paper considers a geometrical inequality for a right triangle. Some generalizations are proposed applying Calculus for the proofs.
Keywords: right triangle, inequality, generalization, function, derivative.

Consider the inequality

(1) $$\frac{a^2(b+c)+b^2(a+c)}{abc} \geq 2+\sqrt{2},$$

where a and b are the catheti and c is the hypotenuse of an arbitrary right triangle, which could be written in the form:

(2) $$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{b}{c}\right) \geq 2+\sqrt{2}.$$

We will try to prove the inequality (2) in a standard manner. We have:

(3) $$(A \geq G): \quad \frac{a}{b}+\frac{b}{a} \geq 2,$$

and from $(a-b)^2 \geq 0$, i.e. from $a^2+b^2 \geq 2ab$ we get:

$$2(a^2+b^2) \geq (a+b)^2, \text{ i.e.}$$

$$2c^2 \geq (a+b)^2.$$

Consequently $$a+b \leq c\sqrt{2}, \text{ i.e.}$$

(4) $$\frac{a}{c}+\frac{b}{c} \leq \sqrt{2}.$$

Now, it is enough to add (3) and (4), thus obtaining (2), i.e. (1).

Note, that the above proof is not simple. In the sequel we give six other proofs.

Proof 1. Applying the obvious equality

$$a^2(b+c)+b^2(a+c) = c\left(a^2+b^2\right)+ab(a+b),$$

It is easy to see, that the given inequality (1) is equivalent to the following inequality:

$$\frac{c(a^2+b^2)+ab(a+b)}{abc} \ge 2+\sqrt{2},$$

or $$c(a-b)^2 \ge \left(c\sqrt{2}-a-b\right)ab = \frac{ab\left[2c^2-(a+b)^2\right]}{c\sqrt{2}+a+b} = \frac{ab(a-b)^2}{c\sqrt{2}+a+b}.$$

In the case $a=b$, i.e. for the isosceles right triangle, the above inequality is true, obviously. In the case $a \ne b$, the above inequality is equivalent to the following one:

$$(c\sqrt{2}+a+b)c \ge ab.$$

Obviously, $c^2 \ge 2ab \quad \left(\Leftrightarrow (a-b)^2 \ge 0\right)$, and it follows, that $c^2 \ge ab$. But $c\sqrt{2}+a+b > c$ and consequently $(c\sqrt{2}+a+b)c > c^2$, which implies, that $(c\sqrt{2}+a+b)c > ab$, q.e.d.

Thus, the proof of the given inequality (1) is done.

Proof 2. Let α and β be the acute angles of a right triangle ΔABC. Because $0<\alpha, \beta<\frac{\pi}{2}$, it is clear, that

(5) $$(1-\sin\alpha)(1-\sin\beta) < 1+\sqrt{2}$$

(because $1-\sin\alpha < 1$ and $1-\sin\beta < 1$).[1*)]

Since $\sin\alpha = \frac{a}{c}$ and $\sin\beta = \frac{b}{c}$, the inequality (5) becomes

$$(1+\sqrt{2})c^2-(c-a)(c-b)>0, \text{ i.e.}$$

(6) $$\sqrt{2}c^2+(a+b)c-ab>0.$$

We have $2c^2 \ge (a+b)^2 \quad \left(\Leftrightarrow (a-b)^2 \ge 0\right)$, i.e. $c\sqrt{2} \ge a+b$, and from here

(7) $$c\sqrt{2}-a-b \ge 0.$$

Multiplying (7) by $\sqrt{2}c^2+(a+b)c-ab$ (which is positive because of (6)), we get

$$(c\sqrt{2}-a-b)\left[\sqrt{2}c^2+(a+b)c-ab\right] \ge 0$$

1 *) It is well-known, that a better inequality holds true: $(1-\sin\alpha)(1-\sin\beta) \le \frac{3}{2}-\sqrt{2}$, with an equality in the case $\alpha=\beta=\frac{\pi}{4}$.

or $$c\left[2c^2-(a+b)^2\right]-abc\sqrt{2}+ab(a+b)\geq 0 .$$

Because of $c^2=a^2+b^2$, the conclusion is, that
$$c(a^2+b^2)+ab(a+b)\geq(2+\sqrt{2})abc.$$

Dividing by *abc*, we get
$$\frac{a^2(b+c)+b^2(a+c)}{abc}\geq 2+\sqrt{2}\text{ , q.e.d.}$$

Thus, the inequality (1) holds true, getting equality iff $a=b$.

Proof 3. Because
$$\frac{a^2(b+c)+b^2(a+c)}{abc}=\frac{(\lambda a)^2(\lambda b+\lambda c)+(\lambda b)^2(\lambda a+\lambda c)}{(\lambda a)(\lambda b)(\lambda c)};\quad (\lambda>0)$$

WLOG we can put $c=1$ $(\Rightarrow a,b<1)$. But $c^2=a^2+b^2$, i.e. $b=\sqrt{1-a^2}$ and the inequality (1) takes the form
$$a^2\left(\sqrt{1-a^2}+1\right)+\left(1-a^2\right)(a+1)\geq\left(2+\sqrt{2}\right)a\sqrt{1-a^2}\text{ , i.e.}$$
$$a(1-a^2)+1\geq(2+\sqrt{2}-a)a\sqrt{1-a^2} .$$

Both sides of the last inequality are positive, and after squaring:
$$2a^6-2a^5(\sqrt{2}+2)+a^4(4\sqrt{2}+3)+a^3(2\sqrt{2}+2)-a^2(4\sqrt{2}-5)+2a+1\geq 0\text{ , i.e.}$$
$$(a\sqrt{2}-1)^2(a^4-2a^3+a^2+2a(\sqrt{2}+1)+1)\geq 0 .$$

Because of $(a\sqrt{2}-1)^2\geq 0$ and
$$a^4-2a^3+a^2+2a(\sqrt{2}+1)+1=(a^2-a)^2+2a(\sqrt{2}+1)+1>0 ,$$
the above inequality holds true and thus the given inequality (1) holds true toot. The equality holds in the case $a\sqrt{2}-1=0$. Since $a=\frac{c}{\sqrt{2}}$ and $b=\sqrt{1-a^2}$, then $a=b=\frac{c}{\sqrt{2}}$, i.e. ΔABC is an isosceles right triangle.

Proof 4. We have

$$T=\frac{a^2(b+c)+b^2(a+c)}{abc}=\frac{a}{c}+\frac{a}{b}+\frac{b}{c}+\frac{b}{a}=\frac{a^2+b^2}{ab}+\frac{a+b}{c}=$$

$$=\frac{a^2+b^2}{ab}+\sqrt{\frac{(a+b)^2}{c^2}}=\frac{a^2+b^2}{ab}+\sqrt{1+\frac{2ab}{a^2+b^2}}.$$

Obviously
$$\frac{a^2+b^2}{ab}\geq 2\ \left(\Leftrightarrow (a-b)^2\geq 0\right).$$

By the substitution $u=\dfrac{a^2+b^2}{ab}$; $u\in[2,+\infty)$, we get

$$T=T(u)=u+\sqrt{1+\frac{2}{u}}\ .$$

Further

$$T'(u)=1-\frac{1}{u^2\sqrt{1+\frac{2}{u}}}\geq 1-\frac{1}{8}\sqrt{2}>0\ ,$$

because of $u\geq 2$. Indeed

$$u^4+2u^3\geq 32,$$

$$u^2\sqrt{1+\frac{2}{u}}\geq 4\sqrt{2},$$

$$\frac{1}{u^2\sqrt{1+\frac{2}{u}}}\leq\frac{1}{4\sqrt{2}},$$

$$-\frac{1}{u^2\sqrt{1+\frac{2}{u}}}\geq -\frac{1}{8}\sqrt{2}$$

$$1-\frac{1}{u^2\sqrt{1+\frac{2}{u}}}\geq 1-\frac{1}{8}\sqrt{2}\ .$$

Therefore, $T'(u)>0$ for all $u\in[2,+\infty)$, which means that the function $T(u)$ is strictly increasing for $u\in[2,+\infty)$. It follows, that $T(u)\geq T(2)=2+\sqrt{2}$. Thus, the given inequality (1) is proved. The equality holds for $u=2$, i.e. when $a=b$.

Proof 5. Using that $\sin\alpha=\frac{a}{c}, \cos\alpha=\frac{b}{c}, \tan\alpha=\frac{a}{b}$ and $cotan\alpha=\frac{b}{a}$, the given inequality (1) (or (2)) takes the form

$$\sin\alpha+\cos\alpha+\tan\alpha+cotan\alpha\geq 2+\sqrt{2}.$$

Consider the function

$$f(x)=\sin x+\cos x+\tan x+cotan x,\quad x\in\left(0,\frac{\pi}{2}\right).$$

We have

$$f'(x)=\cos x-\sin x+\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}, \text{ i.e.}$$

$$f'(x)=(\cos x-\sin x)\left(1-\frac{\sin x+\cos x}{\sin^2 x\cdot\cos^2 x}\right).$$

Further

$$f'(x)=0 \text{ for } \sin x-\cos x=0 \text{ or } 1-\frac{\sin x+\cos x}{\sin^2 x\cdot\cos^2 x}=0.$$

It follows that $\sin x-\cos x=0$, that is $\tan x=1$, i.e. $x=\frac{\pi}{4}$.
We will prove that

$$1-\frac{\sin x+\cos x}{\sin^2 x\cdot\cos^2 x}<0,\quad x\in\left(0,\frac{\pi}{2}\right), \text{ i.e. that}$$

(8)
$$\frac{\sin x+\cos x}{\sin^2 x\cdot\cos^2 x}>1,\quad x\in\left(0,\frac{\pi}{2}\right).$$

Because $x\in\left(0,\frac{\pi}{2}\right)$ we have $0<\sin x<1$ and $0<\cos x<1$. Thus,
$0<\sin x\cdot\cos x<1$, i.e.

(9)
$$0<\sin^2 x\cdot\cos^2 x<1.$$

It follows from $sin\,x = \frac{a}{c}$ and $cos\,x = \frac{b}{c}$, that

$$sin\,x + cos\,x = \frac{a+b}{c} > \frac{c}{c} = 1\,;$$

(applying the triangle inequality $a+b>c$). From here

(10) $$sin\,x + cox > 1\,.$$

By (9) and (10) we deduce the inequality (8). Therefore, $f(x)=0$ for $x=\frac{\pi}{4}$.

We have further

$$f''(x) = -sin\,x - cos\,x + \frac{2\,sin\,x}{cos^3 x} + \frac{2\,cos\,x}{sin^3 x}\,,$$

and

$$f''\left(\frac{\pi}{4}\right) = 8 - \sqrt{2} > 0\,,$$

which means that the function $f(x)$ has minimum.
Therefore,

$$min\,f(x) = f\left(\frac{\pi}{4}\right) = 2+\sqrt{2}\text{, i.e.}$$

$$f(x) \ge f\left(\frac{\pi}{4}\right) = 2+\sqrt{2}\,,\ x\in\left(0,\frac{\pi}{2}\right).$$

Putting $x=\alpha$, we get

$$sin\,\alpha + cos\,\alpha + tan\,\alpha + cotan\alpha \ge 2+\sqrt{2}\text{, i.e.}$$

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \ge 2+\sqrt{2}\text{, q.e.d.}$$

The equality holds only in the case $\alpha = \frac{\pi}{4}$, i.e. when $a=b$ (isosceles right triangle).

Now, we will propose a proof of the generalized inequality (1).

Proof 6. Let a,b,c be the sides of an arbitrary triangle and let $c\ge a$, $c\ge b$. We will prove that in that case the following inequality holds true:

(11) $$\frac{a^2(b+c)+b^2(a+c)}{abc} \geq 2 + cosec\frac{\gamma}{2}\cdot$$

First, we will prove the inequality

(12) $$\frac{c^2}{ab} + \frac{2\sqrt{ab}}{c} \geq 2 - 2\cos\gamma + cosec\frac{\gamma}{2}.$$

By the cosine law for the triangle we have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

and because of $a^2 + b^2 \geq 2ab \quad (A \geq G)$:

$$c^2 \geq 2ab - 2ab\cos\gamma, \text{ i.e.}$$

(13) $$\frac{c}{\sqrt{ab}} \geq \sqrt{2(1-\cos\gamma)}.$$

Since $c = max\{a,b,c\}$, i.e. $\gamma = max\{\alpha,\beta,\gamma\}$ and $\alpha+\beta+\gamma=\pi$, then $\gamma \geq \frac{\pi}{3}$, i.e.

$$\cos\gamma \leq \cos\frac{\pi}{3} = \frac{1}{2}$$

$$2(1-\cos\gamma) \geq 1$$

(14) $$\sqrt{2(1-\cos\gamma)} \geq 1.$$

If $f(x) = x^2 + \frac{2}{x}$, then $f'(x) = 2x - \frac{2}{x^2} = \frac{2}{x^2}\left(x^3-1\right) = \frac{2}{x^2}(x-1)(x^2+x+1) > 0$ and for $x > 1$ the function is monotonically increasing. Applying (13) and (14), we get:

$$f\left(\frac{c}{\sqrt{ab}}\right) \geq f\left(\sqrt{2(1-\cos\gamma)}\right)$$

$$\left(\frac{c}{\sqrt{ab}}\right)^2 + \frac{2\sqrt{ab}}{c} \geq 2(1-\cos\gamma) + \sqrt{\frac{2}{1-\cos\gamma}}$$

$$\frac{c^2}{ab} + \frac{2\sqrt{ab}}{c} \geq 2 - 2\cos\gamma + \frac{1}{\sin\frac{\gamma}{2}}$$

$$\frac{c^2}{ab}+\frac{2\sqrt{ab}}{c}\geq 2-2\cos\gamma+cosec\frac{\gamma}{2}.$$

Thus, the inequality (12) is true. For the proof of the inequality (11) we will use the inequality between the arithmetic and the geometric means $(A\geq G)$ and also the cosine law. We have:

$$\frac{a^2(b+c)+b^2(a+c)}{abc}=\frac{a^2+b^2}{ab}+\frac{a+b}{c}=$$

$$=\frac{c^2}{ab}+2\cos\gamma+\frac{a+b}{c}\geq$$

$$\geq\frac{c^2}{ab}+2\cos\gamma+\frac{2\sqrt{ab}}{c}\geq$$

$$\geq 2-2\cos\gamma+cosec\frac{\gamma}{2}+2\cos\gamma=2+cosec\frac{\gamma}{2},$$

where the equality holds iff $a=b$.

Thus, the inequality (11) is true.

Now, when $\gamma=\frac{\pi}{2}$, i.e. for a right triangle, we get:

$$\frac{a^2(b+c)+b^2(a+c)}{abc}\geq 2+cosec\frac{\pi}{4}\text{, i.e.}$$

$$\frac{a^2(b+c)+b^2(a+c)}{abc}\geq 2+\sqrt{2}\,,$$

which proves that the inequality (1) holds true. Obviously, the inequality (11) is a generalization of (1).

Remark 1. The fact that the function

$$f(x)=x^2+\frac{2}{x},\quad x\geq 1$$

is strictly increasing could be proved in an elementary manner. Let $z>y\geq 1$. We have

$$f(z) > f(y) \Leftrightarrow z^2 + \frac{2}{z} > y^2 + \frac{2}{y} \Leftrightarrow yz^3 + 2y - y^3z - 2z > 0 \Leftrightarrow$$

$$\Leftrightarrow (z-y)\big(yz(y+z)-2\big) > 0 \Leftrightarrow z - y > 0 \Leftrightarrow z > y.$$

Remark 2. By means of Differential Calculus we can to prove one more generalization of the inequality (1), namely:

$$\frac{a^{2t}(b^t + c^t) + b^{2t}(a^t + c^t)}{(abc)^t} \geq \mu_t, \ t \in \mathbb{R}.$$

For $t \geq t_0$, where $t_0 = 0{,}3614177407072...$, take $\mu_t = 2 + 2^{\frac{2-t}{2}}$ and consider the equation $4t \cdot 2^{\frac{t}{2}} + t = 2$.

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