Educational Matters
Научно-методически статии

AN IMPROVEMENT OF THE GERRETSEN'S INEQUALITY FOR NON-OBTUSE TRIANGLES

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Abstract. The inequality $s^2 \geq 2R^2 + 8Rr + 3r^2$ is proved in the paper as a generalization of the Gerretsen's inequality $s^2 \geq r(16R - 5r)$ for non-obtuse triangles.
Keywords: non-obtuse triangle, inequality, generalization

In (Bottema et al., 1969) the following inequality 5.14, p. 53 could be found:

$$24Rr - 12r^2 \leq a^2 + b^2 + c^2 \leq 8R^2 + 4r^2, \qquad (1)$$

where a,b,c are the lenghts of the sides of the triangle ΔABC and R and r are the radius of the circumcircle and the radius of the incircleq respectively. This is **the Gerretsen's[1] inequality** published in the Dutch journal *Nieuw Tijdschr*. Wisk. 41(1953), 1-7. By the well known equality $a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$ we rework (1) in the following form:

$$r(16R - 5r) \leq s^2 \leq 4R^2 + 4Rr + 3r^2, \qquad (2)$$

where s is the semi-perimeter of the triangle ΔABC.

In the mathematical literature on inequalities, this inequality is named after Gerretsen and is known to be **Gerretsen's inequality**.

In (Chirciu, 2015) the following series of inequalities 2.41., p. 22, are given:

$$s^2 \geq 16Rr - 5r^2 \geq 15Rr - 3r^2 \geq 14Rr - r^2 \geq \frac{27}{2}Rr \geq 13Rr + r^2 \geq 12Rr + 3r^2 \geq$$

$$\geq 11Rr + 5r^2 \geq 10Rr + 7r^2 \geq 9Rr + 9r^2 \geq 18Rr + 11r^2 \geq 7Rr + 13r^2 \geq 6Rr + 15r^2 \geq$$

$$\geq 5Rr + 17r^2 \geq 4Rr + 19r^2 \geq 3Rr + 21r^2 \geq 2Rr + 23r^2 \geq Rr + 25r^2 \geq 27r^2.$$

We will prove the inequality for non-obtuse triangle:

$$s^2 \geq 2R^2 + 8Rr + 3r^2 . \qquad (3)$$

This inequality is stronger than the inequality $s^2 \geq r(16R - 5r)$ from (2) because:

$$2R^2 + 8Rr + 3r^2 \geq r(16R - 5r)$$
$$\Leftrightarrow 2R^2 - 8Rr + 8r^2 \geq 0 / : 2$$
$$\Leftrightarrow R^2 - 4Rr + 4r^2 \geq 0$$
$$\Leftrightarrow (R - 2r)^2 \geq 0 .$$

The last is obvious.

It follows, that equalities hold true in (2) and (3) iff $a = b = c$, i. e. in the case of an equilateral triangle.

Proof of inequality (2):

By the Cauchy-Buniakowsky-Schwarz's inequality we have:

$$\left(\left(\sqrt{a\cos\alpha}\right)^2 + \left(\sqrt{b\cos\beta}\right)^2 + \left(\sqrt{c\cos\alpha}\right)^2\right) \cdot \left(\left(\sqrt{\frac{\cos\alpha}{a}}\right)^2 + \left(\sqrt{\frac{\cos\beta}{b}}\right)^2 + \left(\sqrt{\frac{\cos\gamma}{c}}\right)^2\right) \geq (\cos\alpha + \cos\beta + \cos\gamma)^2, \text{ i.e.}$$

$$(a\cos\alpha + b\cos\beta + c\cos\gamma)\left(\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c}\right) \geq (\cos\alpha + \cos\beta + \cos\gamma)^2 . \quad (4)$$

Further we apply the sine law:

$$a\cos\alpha + b\cos\beta + c\cos\gamma = 2R\sin\alpha\cos\alpha + 2R\sin\beta\cos\beta + 2R\sin\gamma\cos\gamma =$$

$$= R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 4R\sin\alpha\sin\beta\sin\gamma =$$

$$= \frac{2R\sin\alpha \cdot 2R\sin\beta}{R}\sin\gamma = \frac{2}{R} \cdot \frac{ab}{2}\sin\gamma = \frac{2F}{R},$$

where F is the area of ΔABC .

Apply the sine law again:

$$\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = \frac{\cos\alpha}{2R\sin\alpha} + \frac{\cos\beta}{2R\sin\beta} + \frac{\cos\gamma}{2R\sin\gamma} =$$

$$= \frac{1}{2R}(ctg\alpha + ctg\beta + ctg\gamma) \text{ and } \cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R} .$$

From (4) it follows, that:

$$\frac{2F}{R} \cdot \frac{1}{2R}(ctg\alpha + ctg\beta + ctg\gamma) \geq \left(1 + \frac{r}{R}\right)^2$$

$$\Leftrightarrow ctg\alpha + ctg\beta + ctg\gamma \geq \frac{(R + r)^2}{F} . \quad (5)$$

In (Mitrinović et al., 1989) the equality (69), p. 58 is given:

$$ctg\alpha + ctg\beta + ctg\gamma = \frac{s^2 - r^2 - 4Rr}{2sr}. \qquad (6)$$

From (5) and (6) we have:

$$\frac{s^2 - r^2 - 4Rr}{2sr} \geq \frac{(R \quad r)}{F}$$

$$\Leftrightarrow \frac{s^2 - r^2 - 4Rr}{2F} \geq \frac{R^2 + 2Rr + r^2}{F}$$

$$\Leftrightarrow s^2 \geq 2R^2 + 8Rr + 3r^2 ,$$

Which is the inequality (3), q.e.d.

Remark 1. The inequality (3) proposed by A. W. Walker in (Walker & Greening, 1972).

Remark 2. In (Mitrinović & Volenec, 1989), p. 58, proofs are given, that $ctg\alpha,\ ctg\beta,\ ctg\gamma$ are the roots of the equality

$$2srt^3 - \left(s^2 - r^2 - 4Rr\right)t^2 + 2srt + \left(2R + r\right)^2 - s^2 = 0 .$$

It follows from here by the Vieta's theorem that:

$$ctg\alpha + ctg\beta + ctg\gamma = \frac{s^2 - r^2 - 4Rr}{2sr} .$$

Remark 3. The following two equalities are well-known (Grozdev, 2005), (Grozdev, 2007):

$$|HI|^2 = 4R^2 + 4Rr + 3r^2 - s^2$$

and

$$|GI|^2 = \frac{1}{9}\left(s^2 + 5r^2 - 16Rr\right),$$

where H, I and G are the orthocentre, the incentre and the centroid of ΔABC, respectively. The relations $|HI| \geq 0$ and $|GI|^2 \geq 0$ give evidence of the Gerretsen's inequality (2).

NOTES

1. J. C. Gerretsen, (1907 – 1983), dutch mathematician

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