(A_2, A_4). With respect to the introduced coordinate system let the abscissae of the points A_1 and A_2 be a_1 and a_2, respectively. It follows from the symmetry with respect to S, that the abscissae of the points A_3 and A_4 are $-a_1$ and $-a_2$, respectively. Use the notation

(1) $$a_1^2 + a_2^2 = 2R^2 .$$

The coordinates of the vertices of the equilateral triangles could be expressed in the following way:

(2) $$\begin{aligned} &P'_{12}\left(\frac{a_1+a_2}{2}, \frac{|a_1-a_2|\sqrt{3}}{2}\right), P''_{12}\left(\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|\sqrt{3}}{2}\right), \\ &P'_{23}\left(\frac{-a_1+a_2}{2}, \frac{|a_1+a_2|\sqrt{3}}{2}\right), P''_{23}\left(\frac{-a_1+a_2}{2}, -\frac{|a_1+a_2|\sqrt{3}}{2}\right), \\ &P'_{34}\left(-\frac{a_1+a_2}{2}, \frac{|a_1-a_2|\sqrt{3}}{2}\right), P''_{34}\left(-\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|\sqrt{3}}{2}\right), \\ &P'_{41}\left(\frac{a_1-a_2}{2}, \frac{|a_1+a_2|\sqrt{3}}{2}\right), P''_{41}\left(\frac{a_1-a_2}{2}, -\frac{|a_1+a_2|\sqrt{3}}{2}\right). \end{aligned}$$

It follows from (1) and (2), that the eight points under consideration are located on the ellipse k, which is determined by the equation

(3) $$k : \frac{x^2}{R^2} + \frac{y^2}{3R^2} = 1 .$$

Thus, the lemma is proven.

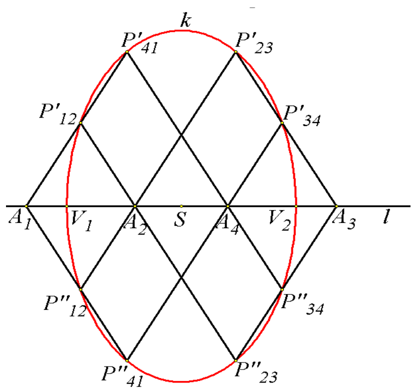

Fig. 1

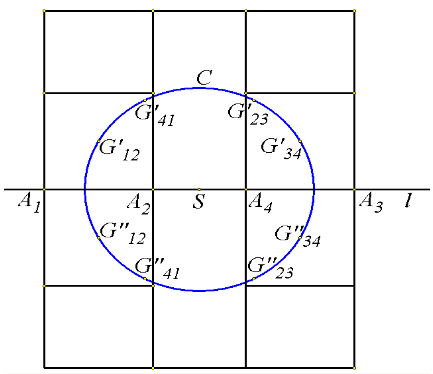

Fig. 2

Lemma 2. *If the point couples* (A_1, A_3) *and* (A_2, A_4) *are located central symmetrically with respect to a point* S *on a line* l*, while the point couples* (G'_{12}, G''_{12}), (G'_{23}, G''_{23}), (G'_{34}, G''_{34}) *and* (G'_{41}, G''_{41}) *are the centers of the squares with bases* A_1A_2, A_2A_3, A_3A_4 *and* A_4A_1*,respectively, then the points* G'_{12}, G''_{12}, G'_{23}, G''_{23}, G'_{34}, G''_{34}, G'_{41} *and* G''_{41} *are located on a circle C with center* S (Fig. 2).

Proof. Consider the same coordinate system as in the proof of lemma 1. The coordinates of the point under consideration could be expressed in the following way:

$$(4)\qquad \begin{aligned} &G'_{12}\left(\frac{a_1+a_2}{2},\frac{|a_1-a_2|}{2}\right), G''_{12}\left(\frac{a_1+a_2}{2},-\frac{|a_1-a_2|}{2}\right),\\ &G'_{23}\left(\frac{-a_1+a_2}{2},\frac{|a_1+a_2|}{2}\right), G''_{23}\left(\frac{-a_1+a_2}{2},-\frac{|a_1+a_2|}{2}\right),\\ &G'_{34}\left(-\frac{a_1+a_2}{2},\frac{|a_1-a_2|}{2}\right), G''_{34}\left(-\frac{a_1+a_2}{2},-\frac{|a_1-a_2|}{2}\right),\\ &G'_{41}\left(\frac{a_1-a_2}{2},\frac{|a_1+a_2|}{2}\right), G''_{41}\left(\frac{a_1-a_2}{2},-\frac{|a_1+a_2|}{2}\right). \end{aligned}$$

It follows easily from (1) and (4), that the eight points are located on the circle C, determined by the equation

$$(5)\qquad C: x^2+y^2=R^2.$$

This ends the proof of the lemma.

It follows from the equations (3) and (5), that the only common points of k and C are the vertices $V_1(-R,0)$ and $V_2(R,0)$ of k.

Thus, we have proved also the following:

Lemma 3. *The circle* C *is tangent to the ellipse* k *at the vertices* V_1 *and* V_2 *of the small axis of* k (Fig. 3).

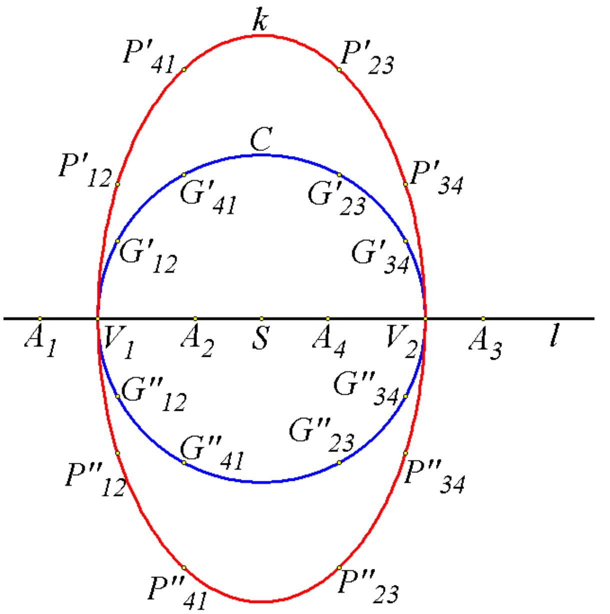

Fig. 3

3. A relation between the roots of the derivative of a polynomial of fourth degree with central symmetric colinear roots and the vertices of the small axis and the center of a special ellipse. We will show now, that the vertices V_1 and V_2 of k define the geometric relation we are looking for between the polynomials that were mentioned at the beginning and their derivatives. More precisely, the following theorem is true:

Theorem. *If a polynomial* $P(z)$ *of fourth degree of a complex variable and with complex coefficients has roots in the point couples* (A_1, A_3) *and* (A_2, A_4)*, which are located central symmetrically with respect to a point* S *on a line* l*, then the derivative* $P'(z)$ *of* $P(z)$ *has roots in the vertices on the small axis and the center* S *of the ellipse* k*, determined by the vertices of the equilateral triangles, which are constructed on the segments* A_1A_2*,* A_2A_3*,* A_3A_4 *and* A_4A_1 (Fig. 4, 5).

Proof. Consider the Gauss coordinate system with a real axis along the line l. This means, that it is possible to use the coordinate system and the notations from the proof of lemma 1. Let now $P(z)$ be a standardized polynomial of fourth degree with roots in the points A_1, A_2, A_3 and A_4. It follows from Vieta's formulae, that $P(z)$ and its derivative $P'(z)$ could be represented in the following way:

(6) $$P(z) = z^4 - 2R^2z^2 + a_1^2a_2^2,$$

(7) $$P'(z) = 4z^3 - 4R^2z.$$

It follows from (7), that the roots of $P'(z)$ are $z_1 = -R$, $z_2 = 0$ and $z_3 = R$, which proves the theorem for a standardized polynomial. Since each polynomial of fourth degree with the mentioned properties could be reduced to a similar form, then the theorem turns out to be true for all polynomials of the kind under consideration.

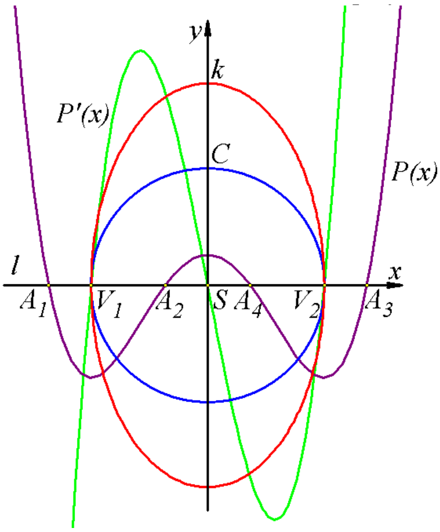

Fig. 4

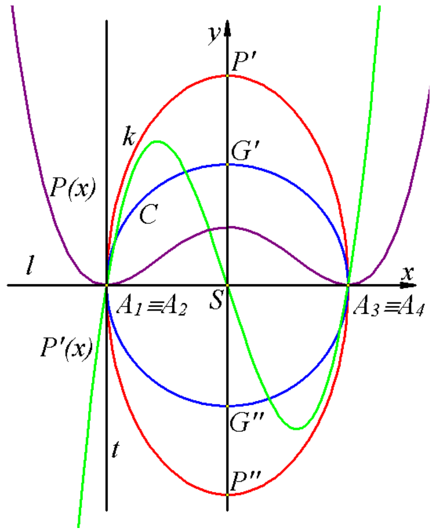

Fig.5

The proven theorem is reasonable in the case $A_1 \equiv A_2$ too. Then $A_3 \equiv A_4$, $P'_{12} \equiv P'_{23} \equiv P'_{34} \equiv P'_{41} \equiv P'$, $P''_{12} \equiv P''_{23} \equiv P''_{34} \equiv P''_{41} \equiv P''$, $G'_{12} \equiv G'_{23} \equiv G'_{34} \equiv G'_{41} \equiv G'$, $G''_{12} \equiv G''_{23} \equiv G''_{34} \equiv G''_{41} \equiv G''$. The circle C is fully determined by its center S and the point A_1 belonging to it. The ellipse k is fully determined by the four points A_1, A_3, P', P'' and the tangent t through A_1 (or A_3), which is perpendicular to l. In this case the vertices of the ellipse on its small axis coincide with the points A_1 and A_3, while the vertices on the big axis are P' and P''. In fact, the polynomial $P(z)$ has double roots in A_1 and A_3, which means that $P'(z)$ has simple roots in the same two points. The cases when $P(z) = P(x)$ is a polynomial with real coefficients of the real variable x, are demonstrated in Fig. 4 and Fig. 5.

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ПОЛИНОМИ ОТ ЧЕТВЪРТА СТЕПЕН С КОЛИНЕАРНИ ЦЕНТРАЛНО СИМЕТРИЧНИ КОРЕНИ

Резюме. Разгледана е геометрична връзка между полином от четвърта степен с корени, разположени симетрично върху права, и корените на неговата производна. За целта е използвана специална елипса, породена от корените на полинома.

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Educational Technologies
Образователни технологии

PROBLEM 6. FROM IMO'2018

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Abstract. The International Mathematical Olympiad is one of the respectable events and one of the most long-lived international educational and scientific competitions. It is the largest, oldest and most prestigious scientific Olympiad for high school students. The 59th edition of the event took place in Cluj-Napoca, Romania, 3 – 14 July 2018. The present paper is dedicated to the sixth problem on the Olympiad paper. A detailed analysis of the problem is proposed in a methodological way, which will be useful for students and teachers in the preparatory process for future participations in mathematical competitions.

Keywords: Olympiad; problem solving

The problem 6 on the paper of the 59th International mathematical Olympiad was solved fully (7 points) by 18 participants, 5 students were marked with 6 points, 2 with 5 points, 5 with 4 points, 11 with 3 points, 26 with 2 points, 108 with 1 point and 419 with 0 points. The mean result of all the 594 participants in the Olympiad from 107 countries is 0. 638, which shows that the problem is hard and needs a detailed analysis.

Problem 6. A convex quadrilateral *ABCD* satisfies $AB.CD = BC.DA$. Point *X* lies inside *ABCD* so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180°$.

Lemma 1. Each convex quadrilateral has a unique interior point *X* such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

Proof: The following cases are possible:

1) If $ABCD$ is a parallelogram, then the point X is the intersection point of the diagonals. This follows from the equality of the cross-opposite angles of the parallelogram. Reversely, the equality of the mentioned angles implies that the point *X* should belong to each of the diagonals and consequently it belongs to both the diagonals simultaneously. Thus the point *X* is unique.