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Образователни технологии

A TRIANGLE AND A TRAPEZOID WITH A COMMON CONIC

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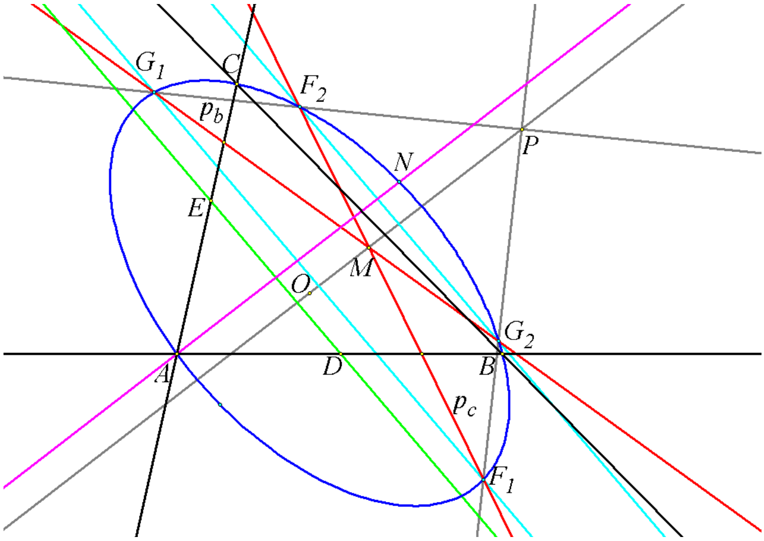
Abstract. The aim of the present note is to propose a generalization of Problem 1 on the IMO'2018 paper. The International Mathematical Olympiad (IMO) is the most prestigious scientific Olympiad for high school students. Its 59th edition took place in Cluj-Napoca, Romania, 3–14 July 2018. The problem 1 on the paper was solved fully (7 points) by 381 participants, 7 students were marked with 6 points, 7 with 5 points, 10 with 4 points, 15 with 3 points, 24 with 2 points, 54 with 1 point and 96 with 0 points. The mean result of all the 594 participants in the Olympiad from 107 countries is 4, 934, which shows that the problem is easy and has not bordered most of the contestants. Nevertheless it turns out to be interesting and originates rich in content ideas.
Keywords: Olympiad; problem solving; triangle; trapezoid; conic

The problem under consideration is the following:

Let Γ be the circumcircle of an acute triangle ABC, while D and E be on the sides AB and AC respectively verifying $AD = AE$. If the perpendicular bisectors of the segments BD and CE meet the small arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ of Γ in points F and G, respectively, prove that the lines DE and FG are parallel or coinciding.

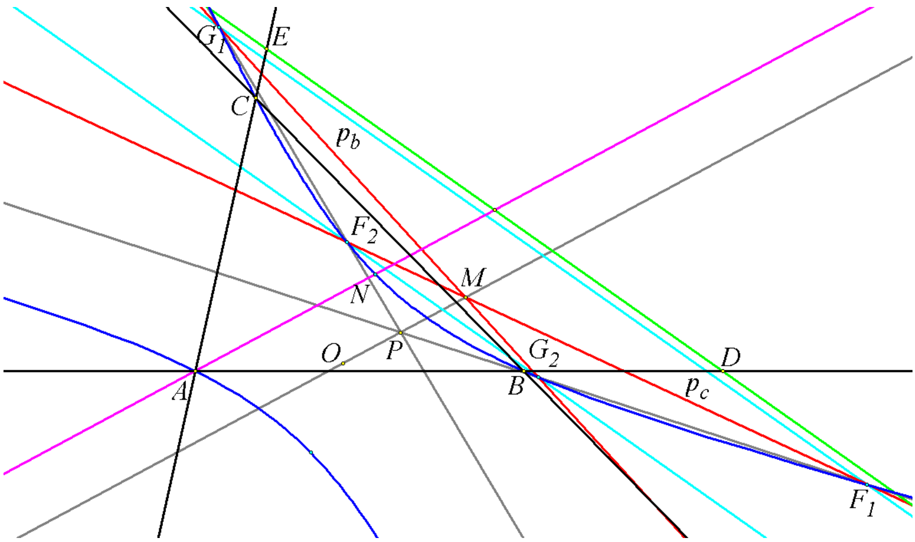
First, replace the circumcircle Γ of ΔABC by an arbitrary conic k with center O. Let D be an arbitrary point on the line AB. For the identification of point E on the line AC, we need an interpretation of the relation $AD = AE$ from the initial problem. The relation shows that ΔDEA is isosceles and that the angular bisector of $\sphericalangle BAC$ is the perpendicular bisector of DE. In other words, the angular bisector passes through the mid-point of DE and is conjugated with DE with respect to Γ. On the other hand, the angular bisector in question crosses the diameter of Γ through the midpoint of BC in a point on Γ. For this reason we consider the following construction: The diameter of k through the midpoint of the side BC meets k in point N. The line through D, which is conjugate with AN, meets AC in point E. Construct the lines p_c and p_b through the midpoints of BD and CE, respectively, to be conjugated with the lines AB and AC. Let the lines p_c and p_b meet k in the point couples (F_1, F_2) and (G_1, G_2), respectively



The following properties of the above construction are satisfied:

A) Two of the lines F_1G_1, F_2G_2, F_1G_2 and F_2G_1 are parallel to DE or one of them coincides with DE;

B) If $F_1G_2 \cap F_2G_1 = P$ and $p_c \cap p_b = M$, then the points O, M and P are on a line which is parallel to AN.



For the proof of the first assertion we will use barycentric coordinates with respect to ΔABC with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, $O(x_0,y_0,z_0)$ $(x_0+y_0+z_0=1)$ and $D(\delta,1-\delta,0)$ (see [1]).

The equation of k is the following

(1) $k:(1-2x_0)x_0yz+(1-2y_0)y_0zx+(1-2z_0)z_0xy=0$.

By this equation and the parametric equations

$x=-2x_0t$, $y=\frac{1}{2}+(1-2y_0)t$, $z=\frac{1}{2}+(1-2z_0)t$

of the diameter of k through the midpoint of BC we find the coordinates of the intersection points N_1 and N_2 in the following form

$$(2)\quad N_1\left(\frac{(1-2y_0)(1-2z_0)-2D_0}{(1-2y_0)(1-2z_0)}x_0,\frac{y_0(1-2z_0)+D_0}{1-2z_0},\frac{z_0(1-2y_0)+D_0}{1-2y_0}\right),$$

$$(3)\quad N_2\left(\frac{(1-2y_0)(1-2z_0)+2D_0}{(1-2y_0)(1-2z_0)}x_0,\frac{y_0(1-2z_0)-D_0}{1-2z_0},\frac{z_0(1-2y_0)-D_0}{1-2y_0}\right),$$

където $D_0=\sqrt{y_0z_0(1-2y_0)(1-2z_0)}$.

We assume that $N\equiv N_1$. The other case could be considered analogously.

Further it will be necessary to determine the coordinates of a vector, which is conjugate with a given vector. It follows from the results in [2], that in case the vector $\vec{v}(v_1,v_2,v_3)$ is conjugate with the vector $\vec{u}(u_1,u_2,u_3)$, then the following equations are satisfied:

$$(4)\quad \begin{aligned} v_1&=(1-2x_0)\left[(y_0-z_0)u_1-x_0u_2+x_0u_3\right],\\ v_2&=(1-2y_0)\left[y_0u_1+(z_0-x_0)u_2-y_0u_3\right],\\ v_3&=(1-2z_0)\left[-z_0u_1+z_0u_2+(x_0-y_0)u_3\right]. \end{aligned}$$

A vector which is collinear with $\overrightarrow{AN}$ has the following coordinates:

$$\begin{aligned} &-(1-2y_0)(1-2z_0)(y_0+z_0)-2x_0D_0,\\ &(1-2y_0)(1-2z_0)y_0+(1-2y_0)D_0,\\ &(1-2y_0)(1-2z_0)z_0+(1-2z_0)D_0. \end{aligned}$$

Applying these coordinares and the formulae (4) we deduce that the vector

$$\left((1-2x_0)(z_0-y_0),2y_0^2-y_0-D_0,-2z_0^2+z_0+D_0\right)$$

is conjugate with $\overrightarrow{AN}$. Consequently the line d, which passes through the point D and is conjugate with the line AN has the following parametric equations:

$$d: x=\delta+(1-2x_0)(z_0-y_0)d_0,\ y=1-\delta+(2y_0^2-y_0-D_0)d_0,\ z=(-2z_0^2+z_0+D_0)d_0.$$

From the last equations we obtain the coordinates of the common point E of d and the line $AC: y=0$:

$$E\left(-\frac{\left[(1-2z_0)z_0+D_0\right]\delta+(1-2x_0)(z_0-y_0)}{2y_0^2-y_0-D_0},0,\frac{(-2z_0^2+z_0+D_0)(\delta-1)}{2y_0^2-y_0-D_0}\right).$$

Further, by the formulae (4) we determine the conjugate directions of the vectors $\overrightarrow{BA}(1,-1,0)$ and $\overrightarrow{AC}(-1,0,1)$. Thus, we obtain the parametric equations of the lines p_c and p_b:

$$p_c: \begin{cases} x=\dfrac{\delta}{2}+(1-2x_0)t_c, \\ y=\dfrac{2-\delta}{2}+(1-2y_0)t_c, \\ z=-2z_0t_c, \end{cases}$$

$$p_b: \begin{cases} x=-\dfrac{\left[(1-2z_0)z_0+D_0\right]\delta+(1-2x_0)(z_0-y_0)}{2(2y_0^2-y_0-D_0)}+(1-2x_0)t_b, \\ y=-2y_0t_b, \\ z=\dfrac{(-2z_0^2+z_0+D_0)\delta+2(y_0^2+z_0^2-D_0)+x_0-1}{2(2y_0^2-y_0-D_0)}+(1-2z_0)t_b. \end{cases}$$

From these equations and (1) we find the coordinates of the points F_1, F_2, G_1 and G_2 in the form:

$$F_1\left(\frac{\delta}{2}-\frac{(1-2x_0)(1-2y_0)+D_c}{2(1-2y_0)},\frac{2-\delta}{2}-\frac{(1-2x_0)(1-2y_0)+D_c}{2(1-2x_0)},\frac{(1-2x_0)(1-2y_0)+D_c}{(1-2x_0)(1-2y_0)}z_0\right),$$

$$F_2\left(\frac{\delta}{2}-\frac{(1-2x_0)(1-2y_0)-D_c}{2(1-2y_0)},\frac{2-\delta}{2}-\frac{(1-2x_0)(1-2y_0)-D_c}{2(1-2x_0)},\frac{(1-2x_0)(1-2y_0)-D_c}{(1-2x_0)(1-2y_0)}z_0\right),$$

$$x_{G_1} = -\frac{\left[\left(1-2z_0\right)z_0 + D_0\right]\delta + \left(1-2x_0\right)\left(z_0 - y_0\right)}{2\left(2y_0^2 - y_0 - D_0\right)} - \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) + D_b}{2\left(1-2z_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

$$y_{G_1} = \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) + D_b}{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

$$z_{G_1} = \frac{\left(-2z_0^2 + z_0 + D_0\right)\delta + 2\left(y_0^2 + z_0^2 - D_0\right) + x_0 - 1}{2\left(2y_0^2 - y_0 - D_0\right)} - \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) + D_b}{2\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

$$x_{G_2} = -\frac{\left[\left(1-2z_0\right)z_0 + D_0\right]\delta + \left(1-2x_0\right)\left(z_0 - y_0\right)}{2\left(2y_0^2 - y_0 - D_0\right)} - \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) - D_b}{2\left(1-2z_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

$$y_{G_2} = \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) - D_b}{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

$$z_{G_2} = \frac{\left(-2z_0^2 + z_0 + D_0\right)\delta + 2\left(y_0^2 + z_0^2 - D_0\right) + x_0 - 1}{2\left(2y_0^2 - y_0 - D_0\right)} - \frac{\left(1-2z_0\right)\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right) - D_b}{2\left(1-2x_0\right)\left(-2y_0^2 + y_0 + D_0\right)},$$

where
$D_c = \left(1-2x_0\right)\left(1-2y_0\right)\left[-\left(1-2z_0\right)\delta^2 + 2\left(1-2z_0\right)\delta + \left(1-2x_0\right)\left(1-2y_0\right)\right]$
and $D_b = \left[z_0\left(1-2z_0\right) + D_0\right]D_c$.

From the coordinates of F_1 and F_2 we obtain the equations

$$x_{G_1} - x_{F_1} = \left(z_0 - y_0\right)\left(1-2x_0\right)m, \quad y_{G_1} - y_{F_1} = \left(2y_0^2 - y_0 - D_0\right)m,$$
$$z_{G_1} - z_{F_1} = \left(-2z_0^2 + z_0 + D_0\right)m,$$

where $m = \dfrac{\left(1-2x_0\right)\left(1-2y_0\right)\left(1-2z_0\right)\left(\delta - 1\right) + \left[\left(1-2y_0\right)\left(1-2z_0\right) + 2D_0\right]D_c}{2\left(1-2x_0\right)\left(1-2y_0\right)\left(1-2z_0\right)\left(2y_0^2 - y_0 - D_0\right)}$.

On the other hand $\overrightarrow{DE}\left(\left(z_0-y_0\right)\left(1-2x_0\right)n,\left(2y_0^2-y_0-D_0\right)n,\left(-2z_0^2+z_0+D_0\right)n\right)$,

where $n=\dfrac{1-\delta}{2y_0^2-y_0-D_0}$. Consequently $F_1G_1 \parallel DE$ or $F_1G_1 \equiv DE$. Analogously it could be shown that $F_2G_2 \parallel DE$ or $F_2G_2 \equiv DE$.

The second assertion from the beginning could be obtained in the following way: Since M and P are the intersection points of the diagonals and the leg-lines of the trapezoid $G_1F_1G_2F_2$, it follows from Steiners's theorem [3] that the line MP passes through the mid-points of the bases G_1F_1 and G_2F_2. Consequently, the line MP is a diameter of k and for this reason it passes through its center O. Additionally the line MP is conjugate with the bases of the trapezoid $G_1F_1G_2F_2$ and the line DE. But the last is conjugate with AN according to the construction. Consequently $MP \parallel AN$.

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ТРИЪГЪЛНИК И ТРАПЕЦ С ОБЩА КРИВА ОТ ВТОРА СТЕПЕН

Резюме. Целта на настоящата бележка е да представи едно обобщение на задача 1 от темата на IMO'2018. Международната олимпиада по математика (IMO) е най-престижната научна олимпиада за гимназиални ученици. Нейното 59-о издание се състоя в Клуж-Напока, Румъния, в периода 3 – 14 юли 2018 г. Задача 1 от темата беше решена пълно (7 точки) от 381 участници, 7 ученици бяха оценени с 6 точки, 7 – с 5 точки, 10 – с 4 точки, 15 – с 3 точки, 24 – с 2

точки, 54 – с 1 точка, и 96 – с 0 точки. Средният резултат на всичките 594 участници в олимпиадата – представители на 107 държави, е 4,934 точки, което показва, че задачата е лесна и не е затруднила по-голямата част от състезателите. Независимо от това се оказва, че задачата е интересна и дава възможност за реализация на съдържателни идеи.

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