Educational Research
Научно–методически статии

A REFINEMENT OF AN INEQUALITY WITH RADICALS

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Abstract. A new algebraic inequality with radicals is proposed in the paper. Its refinement is considered too. The arithmetic-geometric mean (AM-GM) inequality and the Cauchy-Buniakowsky-Schwarz (CBS) inequality are used in the proofs.

Keywords: inequality, refinement, radical, AN-GM inequality, CBS inequality, function, minimum.

1. Introduction.

The following algebraic inequaity is discussed in the sequel:

(1) $$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}<\sqrt{c(ab+1)+1}\,,$$

where $a,b,c\in\mathbb{R}$ and $a,b,c\geq 1$. Four proofs are proposed of a refinement of the inequality (1), namely:

(2) $$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\leq\sqrt{c(ab+1)}\,,$$

where $a,b,c\in\mathbb{R}$ and $a,b,c\geq 1$.

2. Proof of the inequality (1).

Consider the substitutions: $a-1=x^2$, $b-1=y^2$, $c-1=z^2$, $(x,y,z\geq 0)$. Now, the given inequality (1) becomes:

$$x+y+z<\sqrt{\left(z^2+1\right)\left(x^2y^2+x^2+y^2+2\right)+1}\,,$$

and from here after squaring and subtracting we get:

$$x^2y^2z^2+x^2z^2+y^2z^2+x^2y^2+z^2-2xy-2xz-2yz+3>0\,,$$

or $$(xy-1)^2+(yz-1)^2+(xz-1)^2+z^2\left(x^2y^2+1\right)>0\,.$$

This inequality is exact for all $x,y,z\geq 0$, which implies that the given inequality (1) is also exact for all $a,b\geq 1$. Thus, in (1) the strong inequality $<$ is valid. What does that mean? In any case this gives a possibility for the refinement of (1).

3. Refinement of the inequality (1).

We propose four proofs of the inequality (2), i.e. of the refinement of the given inequality (1).

Proof 1. We start by the obvious inequality:

$$\left(\sqrt{x-1}\cdot\sqrt{y-1}-1\right)^2 \geq 0;\ (x,y\geq 1),$$
$$\Leftrightarrow (x-1)(y-1)-2\sqrt{x-1}\cdot\sqrt{y-1}+1\geq 0,$$
$$\Leftrightarrow xy-x-y-2\sqrt{x-1}\cdot\sqrt{y-1}+2\geq 0,$$
$$\Leftrightarrow xy\geq x-1+2\sqrt{x-1}\cdot\sqrt{y-1}+y-1,$$

and from here we get

$$xy\geq\left(\sqrt{x-1}+\sqrt{y-1}\right)^2, \text{ i.e.}$$

(3) $$\sqrt{xy}\geq\sqrt{x-1}+\sqrt{y-1},$$

where the equality holds true if

$$\sqrt{x-1}\cdot\sqrt{y-1}=1, \text{ i.e. if } (x-1)(y-1)=1.$$

From here

(4) $$xy=x+y$$

.

Now, we get now from (3), that:

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\leq\sqrt{ab}+\sqrt{c-1}=\sqrt{(ab+1)-1}+\sqrt{c-1}\overset{(3)}{\leq}\sqrt{c(ab+1)}, \text{ i.e.}$$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\leq\sqrt{c(ab+1)}, \text{ q.e.d.}$$

It follows from (4), that the equality holds true if

$$c(ab+1)=c+ab+1\Rightarrow abc=ab+1.$$

Applying $ab=a+b$ and $abc=ab+1$, we obtain:

(5) $$c=\frac{ab+1}{ab}=1+\frac{1}{ab}=1+\frac{1}{a+b}.$$

Proof 2. Using the the substitution $a-1=x^2,\ b-1=y^2,\ c-1=z^2,\ (x,y,z\geq 0)$ in the inequality (2), we obtain:

$$x+y+z\le\sqrt{\left(z^2+1\right)\left(x^2y^2+x^2+y^2+2\right)}$$

and after squaring and subtracting:

$$(*)\qquad x^2y^2z^2+x^2y^2+x^2z^2+y^2z^2+z^2-2xy-2xz-2yz+2\ge 0,$$

or

$$(6)\qquad \left(x^2y^2+x^2+y^2+1\right)z^2-2\left(x+y\right)z+x^2y^2-2xy+2\ge 0.$$

We have, that $f\left(z\right)=\alpha z^2+\beta z+\gamma\ge 0$ for all $z\in\mathbb{R}$ if $\alpha>0$ and $D=\beta^2-4\alpha\gamma\le 0,\ \left(\alpha,\beta,\gamma\in\mathbb{R}\right)$. Consequently, the inequality (6) holds true if

$$\alpha=x^2y^2+x^2+y^2+1>0 \text{ (which holds true obviously)}$$

and

$$D=4\left[\left(x+y\right)^2-\left(x^2y^2+x^2+y^2+1\right)\left(x^2y^2-2xy+2\right)\right]\le 0$$

or

$$(7)\qquad x^4y^4+x^4y^2+x^2y^4+3x^2y^2+x^2y^2+2\ge 2x^3y^3+2x^3y+2xy^3+4xy.$$

What remains is to prove the inequality (7). Applying the AM-GM inequality, we get:

$$\frac{x^4y^4+x^2y^2}{2}\ge x^3y^3,\ \frac{x^4y^2+x^2}{2}\ge x^3y,\ \frac{x^2y^4+y^2}{2}\ge xy^3,\ \frac{2x^2y^2+2}{2}\ge 2xy,$$

and from here after addition and subtraction, we obtain the inequality (7). Because $\alpha>0$ and $D=\beta^2-4\alpha\gamma\le 0$, it follows, that the inequality (6) is correct, i.e. the given inequality (2) is valid.

Note, that the equality in (7) holds true iff AM = GM, i.e. $xy=1$. This implies that $D=0$ if:

$$\left(2+x^2+\frac{1}{x^2}\right)z^2-2\left(x+\frac{1}{x}\right)+1=0,$$

i.e.

$$\left[\left(x+\frac{1}{x}\right)z-1\right]^2=0.$$

From here

$$z=\frac{1}{x+\frac{1}{x}}=\frac{x}{x^2+1}.$$

Taking into account, that $\alpha-1=x^2$, $b-1=y^2=\frac{1}{x^2}$, $c-1=z^2=\frac{x^2}{\left(x^2+1\right)^2}$, we

conclude, that the equality in (2) holds true iff: $c=\frac{ab+1}{ab}=1+\frac{1}{ab}=1+\frac{1}{a+b}$, which is (5).

Remark. In this way we can prove the inequality $(*)$ too: use the folloowing form of $(*)$:

$$\left(x^2y^2-2xy+1\right)+\left(x^2z^2+y^2z^2+1-2xz-2yz\right)+x^2y^2z^2+z^2\geq 0,$$

or

$$\left(x^2y^2-2xy+1\right)+\left(x^2z^2+y^2z^2+1-2xz-2yz+2xyz^2\right)+\left(x^2y^2z^2+z^2-2xyz^2\right)\geq 0,$$

i.e.

$$(**)\qquad \left(xy-1\right)^2+\left(xz+zy-1\right)^2+z^2\left(xy-1\right)^2\geq 0.$$

The last inequality is true obviously. The equality in (**) holds when $xy-1=0$ and $xz+zy-1=0$, i.e. when $x^2y^2=1$ and $z^2\left(x^2+y^2+2\right)=1$, using the substitution:

$$c=\frac{ab+1}{ab}=1+\frac{1}{ab}=1+\frac{1}{a+b}.$$

Proof 3. We will apply again the substitution $a-1=x^2,\ b-1=y^2,\ c-1=z^2$, where $a,b,c\geq 1$, i.e. $x,y,z\geq 0$. Thus, the given inequality (2) becomes:

$$(8)\qquad x+y+z\leq\sqrt{\left(z^2+1\right)\left[\left(x^2+1\right)\left(y^2+1\right)+1\right]}.$$

The Cauchy-Buniakowsky-Schwarz inequality says for $n=2$, that:

$$\left(a_1b_1+a_2b_2\right)^2\leq\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right);\ \ \left(a_1,a_2,b_1,b_2\in\mathbb{R}\right).$$

From here for $a_1=x,\ b_1=1,\ a_2=1,\ b_2=y$ we get:

$$\left(x\cdot 1+1\cdot y\right)^2\leq\left(x^2+1\right)\left(y^2+1\right),\text{ i.e.}$$

$$(9)\qquad x+y\leq\sqrt{\left(x^2+1\right)\left(y^2+1\right)},$$

For $a_1=z,\ b_1=1,\ a_2=1,\ b_2=\sqrt{\left(x^2+1\right)\left(y^2+1\right)}$ applying the CBS inequality, we obtain:

$$z\cdot 1+1\cdot\sqrt{\left(x^2+1\right)\left(y^2+1\right)}\leq\sqrt{\left(z^2+1\right)\left[1+\left(x^2+1\right)\left(y^2+1\right)\right]},\text{ i.e.}$$

(10) $$z+\sqrt{\left(x^2+1\right)\left(y^2+1\right)}\le\sqrt{\left(z^2+1\right)\left[1+\left(x^2+1\right)\left(y^2+1\right)\right]}.$$

After summing the inequalities (9) and (10), it follows that:

$$x+y+z\le\sqrt{\left(z^2+1\right)\left[1+\left(x^2+1\right)\left(y^2+1\right)\right]},$$

i.e. the inequality (8) holds truet. The equality in (8) holds true iff

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}\Rightarrow\frac{x}{1}=\frac{1}{y}\Rightarrow xy=1\Rightarrow\sqrt{a-1}\cdot\sqrt{b-1}=1\Rightarrow(a-1)(b-1)=1\Rightarrow ab=a+b$$

and from here:

$$\frac{z}{1}=\frac{1}{\sqrt{\left(x^2+1\right)\left(y^2+1\right)}}\Rightarrow z\sqrt{\left(x^2+1\right)\left(y^2+1\right)}=1\Rightarrow$$

$$\Rightarrow z^2\left(x^2+1\right)\left(y^2+1\right)=1\Rightarrow ab(c-1)\Rightarrow abc=1+ab\Rightarrow c=1+\frac{1}{ab},$$

i.e. $ab=a+b$ and $c=1+\frac{1}{ab}$, i.e. $a=1+t,\ b=1+\frac{1}{t}$ and $c=1+\frac{1}{ab};\ (t>0)$.

Proof 4. We will assume, that a and b are constant, while c is one variable. Consider the function $f:[1,+\infty)\to\mathbb{R}$, where

$$f(c)=\sqrt{c(ab+1)}-\sqrt{c-1}-\sqrt{a-1}-\sqrt{b-1}.$$

Because

$$f'(c)=\frac{\sqrt{ab+1}}{2\sqrt{c}}-\frac{1}{2\sqrt{c-1}},$$

it follows that

$$f'(c)=0\Rightarrow\sqrt{ab+1}\cdot\sqrt{c-1}=\sqrt{c}\Rightarrow(ab+1)(c-1)=c\Rightarrow abc-ab+c-1=c\Rightarrow c=1+\frac{1}{ab}.$$

From here

and

$$f''(c)=\frac{1}{4(c-1)\sqrt{c-1}}-\frac{\sqrt{ab+1}}{4c\sqrt{c}},$$

$$f''\left(1+\frac{1}{ab}\right)=\frac{ab\sqrt{ab}}{4}-\frac{ab\sqrt{ab}\cdot\sqrt{ab+1}}{4(ab+1)\sqrt{ab+1}}=\frac{ab\sqrt{ab}}{4}\left(1-\frac{1}{1+ab}\right)=\frac{a^2b^2\sqrt{ab}}{4(1+ab)}>0,$$

Thus concluding, that the function $f(c)$ has minimum for $c = 1 + \frac{1}{ab}$. We have, that $f_{min} = f\left(1 + \frac{1}{ab}\right) = \frac{ab+1}{\sqrt{ab}} - \frac{1}{\sqrt{ab}} - \sqrt{a-1} - \sqrt{b-1} = \sqrt{ab} - \sqrt{a-1} - \sqrt{b-1}$.

Further:

$$\sqrt{ab} \geq \sqrt{a-1} + \sqrt{b-1} \Leftrightarrow ab \geq a + b - 2 + 2\sqrt{(a-1)(b-1)} \Leftrightarrow$$

$$\Leftrightarrow (a-1)(b-1) - 2\sqrt{(a-1)(b-1)} + 1 \geq 0$$

$$\Leftrightarrow \left[\sqrt{(a-1)(b-1)} - 1\right]^2 \geq 0,$$

and we get

$$f_{min} = f\left(1 + \frac{1}{ab}\right) \geq 0.$$

Finally:

$$f(c) \geq f_{min} = f\left(1 + \frac{1}{ab}\right),$$

i.e.

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \sqrt{c(ab+1)},$$

q.e.d.

The equality holds iff $(a-1)(b-1) = 1$, i.e. $b = \frac{a}{a-1}$ and $c = 1 + \frac{1}{ab} = \frac{a^2 + a - 1}{a^2}$, where $a > 1$.

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