A POSSIBILITY TO TEACH AND LEARN MATHEMATICS BY THEATRE TECHNOLOGY

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Abstract. In the recent years new approaches were created to teach and learn Mathematics. One of them is by using theatre plays with mathematical content. The present paper proposes a scenario of such a play based on linear Diophantine equations.
Keywords: linear Diophantine equation; Euclid; Euler; theatre; Little Red Riding Hood; intelligence

Introduction

The main goals of Mathematics education is to prepare students to: communicate and reason mathematically, make connections between Mathematics and its applications, solve problems, become mathematically literate, appreciate and value Mathematics, make informed decisions as contributors to society, etc. Students who have met these goals: gain an understanding and appreciation of the role of Mathematics in society; exhibit a positive attitude toward mathematics; engage in mathematical problem solving; contribute to mathematical discussions; take risks in performing mathematical tasks; exhibit curiosity about Mathematics and situations involving Mathematics. Teachers can assist students in attaining these goals by developing a classroom atmosphere that fosters conceptual understanding through: taking risks, thinking and reflecting independently; sharing and communicating mathematical understanding, solving problems in individual and group projects; pursuing greater understanding of Mathematics, appreciating the value of Mathematics throughout History.

The American Alliance for Theatre education observes in their webpage that: "Drama improves academic performance." Numerous studies have demonstrated a correlation between drama involvement and academic achievement. The College Entrance Examination Board reported student scores using data from the Student Description Questionnaire indicating student involvement in various activities, including the arts. As compared to their peers with no arts coursework or involvement, students involved in drama performance scored an average of 65.5 points higher on the verbal component and 35.5 points higher in the math component of the SAT. Students who took courses in drama study or appreciation scored on average 55 points higher on verbal and 26 points higher on math than their non-arts classmates. In 2005, students involved in drama performance outscored the national average SAT score by 35 points on the verbal portion and 24 points on the math section.

Linear Diophantine Equations with two unknowns

Definition. Let $a, b, c \in \mathbb{Z}$, $ab \neq 0$. The linear equation of the form

$$ax + by = c\,,$$

whose solutions (x, y) are ordered pairs of integers, is called *linear Diophantine equation with two unknowns*.

It is important to note that we look for integer solutions of (1) only. More, we look for all integer solutions of the equation. Take the example $x + y = 13$ and let $y \in \mathbb{Z}$ be arbitrary integer. Then $x = 13 - y$ and the equation has infinitely many solutions. In such a case the solutions are expressed in a *parametric form*: $(x = 13 - t, t)$, where $t \in \mathbb{Z}$ is an arbitrary integer. On the other hand, take $6x + 15y = 13$. This equation has no solution. Indeed, if (x, y) is a solution, then the left side of the equation is divisible by 3, while the right one is not since 3 does not divide 13.

Theorem 1. The linear Diophantine equation (1) has a solution if and only if d divides c, where $d = (a,b)$ is the greatest common divisor (GCD) of a and b.

Proof: Let $d = (a,b)$ and (x_0, y_0) is a solution of (1). Then, $ax_0 + by_0 = c$ and it is obvious that d divides c.

Conversely, let $d = (a,b)$. We will use the well-known Bezout identity (Grozdev, 2007), also known as Bezout lemma. The identity says that there exist integers x and y such that $ax + by = d$. The proof is the sequel.

Without loss of generality assume that d is positive. Consider the set $M = \{xa + yb : x, y \in \mathbb{Z})$ and its subset $M_+ = \{z \in M : z > 0\}$, which is not empty obviously. Since $M_+ \subset \mathbb{N}$, it exists a minimal element $x_1 a + y_1 b > 0$ in M_+. Denote it by m, i.e. $m = x_1 a + y_1 b$. Let $p \in M_+$. Then $p = x_2 a + y_2 b > 0$. We can write $p = mq + r$, where $0 \leq r < m$. Note that:

$$x_2 a + y_2 b = p = mq + r = (x_1 a + y_1 b)q + r$$

and from here we have $r = (x_2 - x_1 q)a + (y_2 - y_1 q)b$. Obviously $r \in M$ and $r \geq 0$. Since $r < m$ and m is minimal, it follows that $r = 0$ and consequently m divides p. We will prove that $m = d$. If $p \in M_+$, we have proved already that m divides p. Since we can consider that a and b are elements of M_+, it follows that m divides a and m divides b, i.e. m is a common divisor of a and b. On the other hand, if m_1 is a common divisor of a and b, then m_1 divides $x_1 a + y_1 b$, i.e. m_1 divides m, which means that m is the GCD of a and b. It follows from the Bezout identity that the representation $x_1 a + y_1 b = d$ is the minimal one.

Going back to the proof of Theorem 1 under consideration, assume that d divides c. Then, it exists $k \in \mathbb{Z}$ such that $c = kd$. On the other hand, the Bezout identity says that there exist integers x_1 and x_2 satisfying $x_1 a + y_1 b = d$. Multiply this equation by k and obtain $x_1 ak + y_1 bk = dk$, i.e. $a(kx_1) + b(ky_1) = c$, which means that (kx_1, ky_1) is a solution of (1).

There are two main methods to solve linear Diophantine equations with two unknowns, namely the Euclid algorithm method and the Euler method.

Euclid algorithm method

The method will be presented by examples.

Problem 1. Solve the Diophantine equation $13x + 32y = 5$.

Solution: Since $a = 13$, $b = 32$ and $(13, 32) = 1$ it follows from Theorem 1 that equation has solutions. According to Euclid algorithm we have:

$$32 = 2.13 + 6$$

$$13 = 2.6 + 1$$

$$6 = 6.1$$

Then, $1 = 13 - 2.6 = 13 - 2(32 - 2.13) = 5.13 - 2.32$, i.e. $5.13 + (-2).32 = 1$. Multiply both sides of the least equality by 5. We obtain $13.25 + 32.(-10) = 5$, which means that $(x, y) = (25, -10)$ is a solution of the equation under consideration. It is easy to check that $(x, y) = (25 + 32t, -10 - 13t)$ satisfies the equation for all $t \in \mathbb{Z}$.

Theorem 2. If $d = (a, b)$, d divides c and (x_0, y_0) is a solution of (1), then all solutions are given by $x = x_0 + \frac{b}{d}t$, $y = y_0 - \frac{a}{d}t$, $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

Proof: The direct checking shows that:

$$ax+by=a\left(x_0+\frac{b}{d}t\right)+b\left(y_0-\frac{a}{d}t\right)=c,$$

which means that $x=x_0+\frac{b}{d}t$, $y=y_0-\frac{a}{d}t$ is a solution for all $t=0,\ \pm1,\ \pm2,\ \pm3, \ldots$ We will prove that each solution can be represented in this form.

Let (x,y) be a solution. We have

$ax+by=ax_0+by_0 \Leftrightarrow a(x-x_0)=b(y_0-y) \Leftrightarrow \frac{a}{d}(x-x_0)=\frac{b}{d}(y_0-y)$.

Since $d=(a,b)$, then $\left(\frac{a}{d},\frac{b}{d}\right)=1$ and it follows from the last equation that $\frac{b}{d}$ divides $(x-x_0)$ and $\frac{a}{d}$ divides (y_0-y). Thus, $x-x_0=\frac{b}{d}u$ and $y_0-y=\frac{a}{d}v$ for some integers *u* and *v*. The equation $a(x-x_0)=b(y_0-y)$ implies, that $u=v$ and this ends the proof.

Problem 2. Solve the Diophantine equation $69x+111y=9000$.

Solution: Since $(69,111)=3$, we divide both sides of the equation by 3 and come to its equivalent form $23x+37y=3000$. Applying the Euclid algorithm method, we find that $23.(-8)+37.5=1$. Further multiply the last equality by 3000 to obtain $23.(-24000)+37.15000=3000$ .Thus, $(x_0,y_0)=(-24000,15000)$. Theorem 2 gives that all solutions of the equation are $x=-24000+37t$, $y=15000-23t$, $t=0,\ \pm1,\ \pm2,\ \pm3, \ldots$

Euler method

This method will be presented by examples too.

Problem 3. Solve the Diophantine equation $738x+621y=45$.

Solution: Let (x,y) be a solution. Since $621<738$, we express *y* by means of *x*, taking into account that $738=1.621+117$ and $45=0.621+45$. It follows that

$$y=\frac{-738x+45}{621}=-x+\frac{-117x+45}{621}.$$

The conclusion is that the number $\frac{-117x+45}{621}=t$ is integer and from here $621t+117x=45$, which is a new Diophantine equation but with smaller coefficients. Proceeding in a similar way we express *x* by means of *t* since $117<621$. Taking into account that $621=5.117+36$ and $45=0.117+45$, it follows that

$$x = \frac{-621t + 45}{117} = -5t + \frac{-36t + 45}{117}.$$

Thus, the number $\frac{-36t + 45}{117} = u$ is integer and $117u + 36t = 45$ is a new Diophantine equation with smaller coefficients. Further, we express t by means of u and taking into account that $117 = 3.36 + 9$, $45 = 1.36 + 9$, it follows that

$$t = \frac{-117u + 45}{36} = -3u + 1 + \frac{-9u + 9}{36} = -3u + 1 + \frac{-u + 1}{4}.$$

The number $\frac{-u + 1}{4} = v$ is integer and $4v + u = 1$. We have $u = -4v + 1$ and plugging backwards we obtain:

$$t = -3u + 1 + v = -3(-4v + 1) + 1 + v = 13v - 2$$
$$y = -x + t = 69v - 11 + 13v - 2 = 82v - 13.$$

The last two equations give the parametric representation of all solutions, namely:

$$x = -69v + 11,\ y = 82v - 13,\ v = 0,\ \pm 1,\ \pm 2,\ \pm 3, ...$$

Problem 4. You have two hour-glasses – the first one measuring 11 minutes and the other one measuring 7 minutes. Is it possible to measure 15 minutes with them?

Solution: Denote by x and y the number of times you use the first our-glass and the second one respectively. Then we have $11x + 7y = 15$, which is a linear Diophantine equation with two unknowns. It has solutions because $(11, 7) = 1$. Solve it by Euler method, for example:

$$y = \frac{-11x + 15}{7} = -x + 2 + \frac{-4x + 1}{7} = -x + 2 + t$$
$$\frac{-4x + 1}{7} = t \Leftrightarrow 7t + 4x = 1$$
$$x = \frac{-7t + 1}{4} = -t + \frac{-3t + 1}{4} = -t + u$$
$$\frac{-3t + 1}{4} = u \Leftrightarrow 4u + 3t = 1$$
$$t = \frac{-4u + 1}{3} = -u + \frac{-u + 1}{3} = -u + v$$

$$\frac{-u+1}{3}=v \Leftrightarrow u=-3v+1.$$

Backwards:

$$t=-u+v=3v-1+v=4v-1$$

$$x=-t+u=-4v+1-3v+1=-7v+2$$

$$y=-x+2+t=7v-2+2+4v-1=11v-1.$$

All solutions of the equation under consideration are $x=-7v+2$, $y=11v-1$, $v\in\mathbb{Z}$. Take $v=0$. Then $x=2$ and $y=-1$. This means the following (especially the negative number of uses):

Start both hour-glasses. The second one will measure 7 minutes. From this moment on we can measure exactly $11-7=4$ minutes, i.e. after 4 minutes the sand in the first hour-glass will flow out. Thus, we can measure 4 minutes. Then you turn upside down the first hour-glass and it will measure 11 minutes more. Here you are, the problem is solved, because $11+4=15$ minutes.

First part of the scenario

We are ready to start a theatre play:

The Little Red Riding Hood jumped out of bed and ran to the kitchen to her Grandma:

– Grandma, I want a boiled egg!

– O key, my darling, but to prepare it according to your favorite taste for eggs, I have to boil it exactly 15 minutes. Unfortunately, the clock stopped and I cannot measure 15 minutes. We have to wait for your Grandpa, who went to buy a new battery for the clock.

– But Grandma, why don't you use the hour-glass?

– I can't, because it measures 7 minutes. I cannot use your Grandpa's hour-glass either, because it measures 11 minutes.

– O, Grandma, look how we will obtain 15 minutes. Start both hour-glasses….

And the story continues with the solution of Problem 4.

Problem 5. You have two pails – the first one is 14-litre, while the second one is 8-litre. Is is possible to measure 4 litres exactly?

Solution: Denote by x and y the number of times you use the first pail and the second one respectively. Then we have $14x+8y=4$, which is a linear Diophantine equation with two unknowns. It has solutions because

$(14,8)=2$ and 2 divides 4. Dividing both sides by 2, we obtain an equivalent form of the equation, namely $7x+4y=2$. Solve this equation by Euler method again:

$$y=\frac{-7x+2}{4}=-x+\frac{-3x+2}{4}=-x+t$$

$$\frac{-3x+2}{4}=t \Leftrightarrow 4t+3x=2$$

$$x=\frac{-4t+2}{3}=-t+\frac{-t+2}{3}=-t+u$$

$$\frac{-t+2}{3}=u \Leftrightarrow t=-3u+2\,.$$

Backwards:

$$x=-t+u=3u-2+u=4u-2$$

$$y=-x+t=-4u+2-3u+2=-7u+4\,.$$

Thus, all solutions of the equation are $x=4u-2$, $y=-7u+4$, $u\in\mathbb{Z}$. Take $u=0$. Then, $x=-2$ and $y=4$. This means the following (especially the negative number of uses):

Fill in the 14-litre pail with water from the tap. Then, fill in the second pail with water from the 14-litre one. Since the second pail is 8-litre, the remaining water in the first one will be $14-8=6$ litres. Further, flow out the second pail and pour the 6 litres from the first one into it. Again, fill in the 14-litre pail brimfuly with water from the tap. What is the situation: we have 14 litres in the first pail and 6 litres in the second one. Now, you can add exactly $8-6=2$ litres to the second pail using water from the first one. The result is $14-2=12$ litres in the first pail and 8 litres in the second one. It remains to pour out the second pail and to fill in brimfuly with water from the first pail. Thus, we have $12-8=4$ litres in the first pail and here are your 4 litres.

Second part of the scenario

We are ready to continue the story:

The Grandpa entered the kitchen.

– Let me see now what a mathematician you are! – he said and showed the two pails he carried with him. – The first pail is 8-litre , while the second one is 14-litre. I have to measure 4 litres exactly and I don’t know how to proceed.

– O, Grandpa, this task is very easy! Come on, fill in the 14-litre pail with water from the tap.

The Grandpa filled in the pail and turned his face to the Little Red Riding Hood impatiently...

And the story continues with the solution of Problem 5.

The whole scenario

The Little Red Riding Hood and Diophantine Equations of First Order

(for students, age 9 – 13)

(*The sketch is performed by one student (monologue), namely the Little Red Riding Hood, who wears a red hat. She plays three roles simultaneously: except her, she performs the role of her grandmother and the role of her grandfather. The roles are changing by means of hats. As grandmother, the Little Red Riding Hood wears an old woman's hat and as grandfather she wears a cap. The performance is in front of a table with two different hour-glasses and two different pails on it.*)

The Little Red Riding Hood jumped out of bed and ran to the kitchen to her Grandma:

– Grandma, I want a boiled egg!

– (*The Little Red Riding Hood puts on the old woman's hat*) O key, my darling, but to prepare it as you have a taste for eggs, I have to boil it exactly 15 minutes. Unfortunately, the clock stopped and I cannot measure 15 minutes. We have to wait for your Grandpa, who went to buy a new battery for the clock.

– (*The Little Red Riding Hood takes off the old woman's hat*) But Grandma, why don't you use the hour-glass?

– (*The Little Red Riding Hood puts on the old woman's hat*) I can't, because it measures 7 minutes. I cannot use your Grandpa's hour-glass either, because it measures 11 minutes.

– (*The Little Red Riding Hood takes off the old woman's hat*) O, Grandma, look how we will obtain 15 minutes. Start both hour-glasses. When the first one will measure 7 minutes, you will put the egg boiling. Since $11-7=4$, after exactly 4 minutes the sand in the second hour-glass will flow out. Thus, we can measure 4 minutes. Then you will turn upside down the second hour-glass and it will measure 11 minutes more. Here you are, the problem is solved, because $11+4=15$ minutes.

Delighted with the mathematical skills of her grandchild, the Grandma took in hand the proposal. Soon the egg was boiled in 15 minutes exactly and the Little Red Riding Hood ate it with satisfaction. She just licked clean, when her Grandpa entered the kitchen.

– (*The Little Red Riding Hood puts on the cap*) Let me see now what a mathematician you are! – he said and showed the two pails he carried with him. – The first pail is 8-litre, while the second one is 14-litre. I have to measure 4 litres exactly and I don't know how to proceed.

– (*The Little Red Riding Hood takes off the cap*) O, Grandpa, this task is very easy! Come on, fill in the 14-litre pail with water from the tap.

The Grandpa filled in the pail and turned his face to the Little Red Riding Hood impatiently.

– And now, Grandpa, fill in the second pail with water from the 14-litre one. Since the second pail is 8-litre, the remaining water in the first one will be $14-8=6$ litres. Further, flow out the second pail and pour the 6 litres from the first one into it. Again, fill in the 14-litre pail brimfuly with water from the tap. What is the situation: we have 14 litres in the first pail and 6 litres in the second one. Now, you can add exactly $8-6=2$ litres to the second pail using water from the first one. The result is $14-2=12$ litres in the first pail and 8 litres in the second one. Grandpa, it remains to pour out the second pail and to fill in brimfuly with water from the first pail. Thus, we have $12-8=4$ litres in the first pail and here are your 4 litres.

– (*The Little Red Riding Hood puts on the cap*) Congratulations, my grandchild, where do you know these things from?

– (*The Little Red Riding Hood takes off the cap*) Grandpa, note, that $14.2-8.3=28-24=4$. This means that we have filled in the 14-litre pail twice, while we have filled in the 8-litre pail three times. We have proceeded in such a way but in a suitable succession. Earlier I have shown to Grandma how to measure 15 minutes by means of the two hour-glasses, the first one measuring 7 minutes and the second one measuring 11 minutes. I have used that $2.11-7=22-7=15$. This means that the 11-minute hour glass has been used twice, while the 7-minute one – only once. At first glance the two tasks are different but actually the principal is one and the same. The general problem is to find integers x and y, verifying $ax+by=c$, where a, b and c are given integers. This equality is known to be Diophantine equation, which we have studied last week in the mathematical circle. In the case of the pails the Diophantine equation is $14x+8y=4$, while in the case of the hour-glasses the equation is $11x+7y=15$. In both cases the Diophantine equations have solutions. In the first case it is so, because the greatest common divisor of 14 and 8 is 2, and 2 divides 4. In the second case the integers 11 and 7 are co-prime, which means that their greatest common divisor is 1. At the same time 1 divides each integer and it divides 15 in particular. The teacher in the circle told us that the Diophantine equation under consideration has solution if and only if the greatest common divisor of a and b divides c. It follows that the Diophantine equation $14x+8y=3$ has no solution, because the greatest common divisor

2 of 14 and 8 does not divide 3. For this reason, Grandpa, if you had asked me to measure 3 litres using your two pails, I would answer that this was not possible. And this is the truth.

Analysis

– Math Topic – Diophantine equations

– Age group 9 – 13

– Knowledge background

– Background Needed – Linear Diophantine equations with two variables, common divisor, prime number, co-prime numbers

– Knowledge acquired – ability of modeling, how to check the existence of a solution of a linear Diophantine equation with two variables

– Skills acquired

– Analytical Thinking – The analysis of the mathematical problem into its constituent parts, finding the common divisors or checking whether two numbers are co-prime provides the necessary evidence for development of analytical thinking skills.

– Mathematical modeling – a real life problem should be translated to a mathematical problem, find the mathematical solution and translate it back to the real life solution.

– Problem solving – starting to solve the problem one should comprehend the conditions and plan the solution.

– Communication – skill of presenting a mathematical idea (Mathematics communication).

In 1983, Howard Gardner, at that time Professor at Harvard University, released his book "Frames of mind" in which he developed his theory of multiple intelligences. He suggests that each person has several types of intelligence, for which he or she naturally display's more or less competence. There are eight of them:

1. Intrapersonal intelligence (self smart);
2. Interpersonal intelligence (people smart);
3. Musical-Rhythmic intelligence (music smart;)
4. Bodily-Kinesthetic intelligence (body smart);
5. Visual-Spatial intelligence (picture smart);
6. Naturalist intelligence (nature smart);
7. Linguistic intelligence (word smart);
8. Logical-mathematical intelligence (number smart).

Traditional mathematics lessons bring logical-mathematic intelligence into play, such as the ability to reason in the geometrical or numerical area, as well the ability to calculate and handle figures, numbers, and geometrical shapes. The other forms or types of intelligence are often casted aside or are completely forgotten. Pedagogically, mixing theatre and Mathematics allows us to solicit almost all the different types of intelligence:

Logical-mathematics: The mathematical content, worked in the classroom and processed in the play, might be reinforced after the theatrical activities. Moreover, these skills are also required in the elaboration of the script, the play. **Spatial**: Recognition of the notion of space in the staging of the play. The movement of students themselves during the play and the recognition of their own position in the space, as well as the position of their fellow students. **Kinaesthetic**: When students are acting, they represent a character or a mathematical symbol. The notion is imprinted in their minds through the movements enacted by their own bodies. **Linguistic**: The work begins with the writing of a script or with the study of a script. In all cases, the language is a method of communication and therefore must be worked, adapted to the audience and perfected, as it is the basis of the play. **Interpersonal**: The relationship between the student and the teacher. The discussions between the students during the development of the script, the elaboration of the play, the feedback of the activity and the work in a group in general, improve communication. **Musical**: Music smarts can be cultivated in a musical, or if there is music or songs in the play. Moreover, during the play, musicality is present in the modulations of the voice, the volume, the rhythm and the speed of speech, which are necessary for the clarity and the pleasantness of the play. **Naturalist**: The decor can make students imagine they are in meadows, near the sea or in a forest. Everything is in their imagination and theatre allows that. **Moreover, even more important** ... the pleasure, the game!

Conclusions

So, what qualifications are needed to begin this theatre practice? It is certainly an advantage if the trainer had an experience in theatre, but it is not necessarily a requirement. Most people have seen at least one play or have read a script. It is not so difficult for teachers to become actors or stage directors, as in a sense they emulate the experience when they enter the classroom. They have their public, and they must convince their audience using rhetoric, body language, etc. Just like the way that famous mathematicians, thinkers or philosophers have done for centuries in the past. The role of the teacher is to create a fun atmosphere beneficial to the game, to reassure the learners and to encourage their participation. The teacher needs to install a sense of mutual respect, to establish a non-judgmental atmosphere where humility and collectivism are necessary, as well as allowing imagination to thrive. It is possible to set up a theatrical activity in the mathematics lesson in different ways depending on the objectives, but also depending on the number of sessions the teacher chooses to use for the work.

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ВЪЗМОЖНОСТ ЗА ПРЕПОДАВАНЕ И ИЗУЧАВАНЕ НА МАТЕМАТИКА ЧРЕЗ ТЕАТЪР

Резюме. През последните години бяха създадени нови подходи за преподаване и изучаване на математика. Един от тях е с използване на театрални пиеси с математическо съдържание. Настоящата статия предлага сценарий за такава пиеса на базата на линейни диофантови уравнения.

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