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A NOTE ON A GENERALIZED DYNAMICAL SYSTEM OCCURS IN MODELLING "THE BATTLE OF THE SEXES": CHAOS IN SOCIOBIOLOGY

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Abstract. In this paper, we propose a new system that occurs in modelling "the battle of the sexes" in evolutionary biology (Hofbauer & Sigmund 1988). The existence of a heteroclinic cycle and a continuous family of periodic orbits of the system is established; then the dynamical characteristics of a time-periodic perturbation of the system are investigated. By using the well-known Melnikov's method, a sufficient condition is obtained for the perturbed system to have a transverse hetero-clinic cycle and hence to possess chaotic behaviour in the sense of Smale. One possible application that Melnikov functions may find in the modelling and synthesis of radiating antenna patterns is considered. We demonstrate some modules for investigating the dynamics of the proposed model. This will be included as an integral part of a planned much more general Web-based application for scientific computing. The proposed new extended model contains many free parameters (the coefficients $a_i, i = 1, 2, \ldots, N$), which makes it attractive for use in the fields of biological applications, chemistry, sociology, lifetime analysis, reaction kinetics, biostatistics, population dynamic, medical research, games theory etc. Finally, a special case of subharmonic solutions is discussed.

Keywords: new system occurs in modelling "the battle of the sexes"; hetero-clinic orbit; Melnikov function; subharmonic Melnikov function; chaotic behavior

1. Introduction

A number of authors devote their research to the classical differential system that occurs in modelling "the battle of the sexes" in evolutionary biology (Hofbauer & Sigmund 1988; Dawkins 1976; Foster & Young 1990; Schuster & Sigmund 1981; Taylor & Jonker 1978). The publications on this topic are significant and varied. It is known that the dynamical game theoretic mating behaviour of males and females can be modelled by a planar system of autonomous ordinary differential equations. This system occurs in modelling "the battle of the sexes" in evolutionary biology. In (Christie et al. 1995) the authors presented the following model

$$\begin{cases} \dfrac{dx}{dt} = x(1-x)(\alpha - \beta y) + \epsilon\,(ax + a_1 \sin(\omega t)) \\ \dfrac{dy}{dt} = y(1-y)(\gamma - \delta x) + \epsilon\,(ay + a_1 \sin(\omega t)) \end{cases} \tag{1}$$

where $0 \leq \epsilon < 1$.

The unperturbed differential system is non-Hamiltonian under general assumptions, becoming Hamiltonian only for $\{\alpha,\ \beta = 2\alpha,\ \gamma = -\alpha,\ \delta = -2\alpha\}$.

The existence of a hetero-clinic cycle and a continuous family of periodic orbits of the system (1) is established. In the serious studies (Hofbauer & Sigmund 1988; Dawkins 1976; Foster & Young 1990; Schuster & Sigmund 1981; Taylor & Jonker 1978; Christie et al. 1995) cited above, the reader can find a considerable volume of literature devoted to this classic model. The reader can find important questions related to the topic Strategies and stability: an opening in game dynamics, some aspects of sociobiology, evolutionarily stable strategies, game dynamics and asymmetric conflicts in the monograph (Hofbauer & Sigmund 1988). The genetic model for the "battle of the sexes" is due to (Smith & Hofbauer 1987).

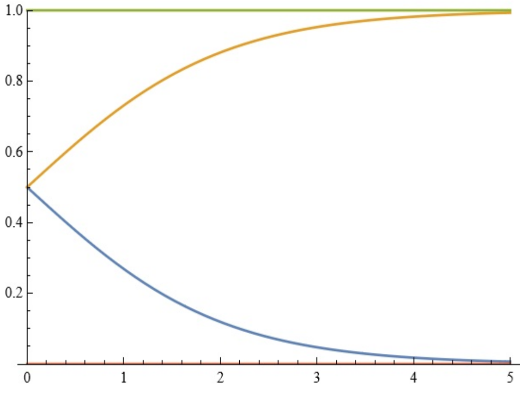

Figure 1. Part of orbits $q_0(t) = (x_0(t), y_0(t))$

In this paper, we suggest a new class of extended asymmetric models of the type (1) that occurs in modelling "he battle of the sexes" in evolutionary biology. Investigations in the light of Melnikov's approach is considered. A sufficient condition is obtained for the perturbed system to have a transverse heteroclinic cycle and hence to possess chaotic behaviour in the sense of Smale. Several simulations are composed. We demonstrate some modules for investigating the dynamics of the proposed model. This will be included as an integral part of a planned much more general Web-based application for scientific computing.

The plan of the paper is as follows. We state our model in Section 2. Investigations in the light of Melnikov's approach is considered in Section 3. Some simulations are presented in Section 4. One possible application that Melnikov functions may find in the modelling and synthesis of radiating antenna patterns is considered in Section 5. We conclude by Section 6.

2. The new model

In this paper, we suggest a modified model of the type:

$$\begin{cases} \dfrac{dx}{dt} &= x(1-x)(\alpha-\beta y)+\epsilon\left(ax+\displaystyle\sum_{j=1}^{N} a_j\sin(j\omega t)\right) \\ \dfrac{dy}{dt} &= y(1-y)(\gamma-\delta x)+\epsilon\left(ay+\displaystyle\sum_{j=1}^{N} a_j\sin(j\omega t)\right) \end{cases} \tag{2}$$

where $0 \le \epsilon < 1$, $a_i \ge 0$; $i = 1, 2, \ldots, N$ and N is integer.

3. Considerations in the light of Melnikov's approach

The system (2) is of the form:

$$\begin{aligned} \frac{dx}{dt} &= f_1(x,y)+\epsilon g_1(x,y,t) \\ \frac{dy}{dt} &= f_2(x,y)+\epsilon g_2(x,y,t). \end{aligned}$$

Let $q_0(t) = (x_0(t), y_0(t))$ be one of the hetero-clinic solutions of the first integral of (2) ($\epsilon = 0$) given by (see for more details precise considerations in (Christie et al. 1995))

$$\begin{aligned} OA:&\ y_0(t)=0;\ x_0(t)=\frac{e^{\frac{\alpha t}{2}}}{2cosh(\frac{\alpha t}{2})}, \\ AB:&\ x_0(t)=1;\ y_0(t)=\frac{e^{\frac{(\gamma-\delta)t}{2}}}{2cosh(\frac{(\gamma-\delta)t}{2})}, \\ BC:&\ y_0(t)=1;\ x_0(t)=\frac{e^{\frac{(\alpha-\beta)t}{2}}}{2cosh(\frac{(\alpha-\beta)t}{2})}, \\ CO:&\ x_0(t)=0;\ y_0(t)=\frac{e^{\frac{\gamma t}{2}}}{2cosh(\frac{\gamma t}{2})}. \end{aligned} \tag{3}$$

(Here $O(0, 0), A(1, 0), B(1, 1), C(1, 1)$ are the other fixed points.)

See also fig. 1 for fixed $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$.

The Melnikov function (Melnikov 1963) corresponding to the system (2) is of the form:

$$M(t_0) = \int_{-\infty}^{\infty} f(q_0(t)) \wedge g(q_0(t), t + t_0) e^{-\int_0^t traceDf(q_0(s))ds} dt$$

$$= \int_{-\infty}^{\infty} \left(x_0(t)(1 - x_0(t))(\alpha - \beta y_0(t)) \left(a y_0(t) + \sum_{j=1}^{N} a_j sin(j\omega(t + t_0)) \right) \right.$$

$$\left. - y_0(t)(1 - y_0(t))(\gamma - \delta x_0(t)) \left(a x_0(t) + \sum_{j=1}^{N} a_j sin(j\omega(t + t_0)) \right) \right) B(t) dt \qquad (4)$$

where

$$B(t) = e^{-\int_0^t ((1 - 2x_0(s))(\alpha - \beta y_0(s)) + (1 - 2y_0(s))(\gamma - \delta x_0(s)))\, ds}.$$

The Melnikov function can be represented in another way, using the following familiar equality

$$sin(i\omega(t + t_0)) = sin(i\omega t)cos(i\omega t_0) + cos(i\omega t)sin(i\omega t_0).$$

We will not dwell on this question here.

For us, it is more important to note that such a representation is more appropriate and can be used directly by users of the corresponding specialized module implemented for example in CAS Mathematica.

If $M(t_0) = 0$ and $\frac{M(t_0)}{dt_0} \neq 0$ for some t_0 and some sets of parameters, then chaos occurs.

The Melnikov function gives a measure of the leading order distance between the stable and unstable manifolds when $\epsilon \neq 0$.

3.1. The case $N = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$

From (4) we find:

$$M(t_0)|_{OA} = a_1 \pi\omega csch(\pi\omega) sin(\omega t_0);$$
$$M(t_0)|_{AB} = -a - a_1 \pi\omega csch(\pi\omega) sin(\omega t_0);$$
$$M(t_0)|_{BC} = -a - a_1 \pi\omega csch(\pi\omega) sin(\omega t_0);$$
$$M(t_0)|_{CO} = a_1 \pi\omega csch(\pi\omega) sin(\omega t_0).$$

Proposition 1. If $N = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, then the roots of the Melnikov function $M(t_0)$ are given as solutions of the equation

$$M(t_0) = a + a_1 \pi\omega csch(\pi\omega) sin(\omega t_0) = 0.$$

The Melnikov condition for existence of chaotic behavior in the sense of Smale in the system (2) for sufficiently small ϵ is

$$\left|\frac{a}{a_1}\right| < \pi\omega csch(\pi\omega).$$

Remark. This result matches the estimate obtained in (Christie et al. 1995).

For example for $N = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, $\omega = 0.78$, $a = 0.11$, $a_1 = 0.1$ the Melnikov function $M(t_0) = 0$ has no roots (fig. 2).

For $N = 1$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, $\omega = 0.68$, $a = 0.12$, $a_1 = 0.258$, $M(t_0) = 0$ has root $t_0 \approx 6.95$ (with multiplicity two) (fig. 3).

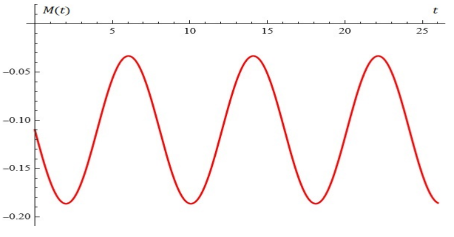

Figure 2

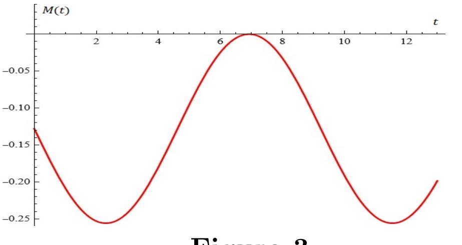

Figure 3

3.2. The case $N = 2$***,*** $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$

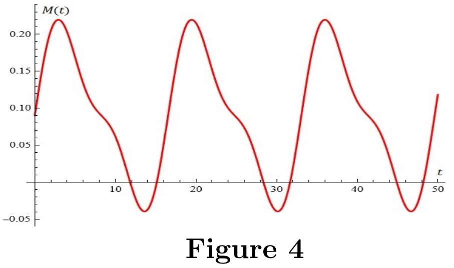

Figure 4

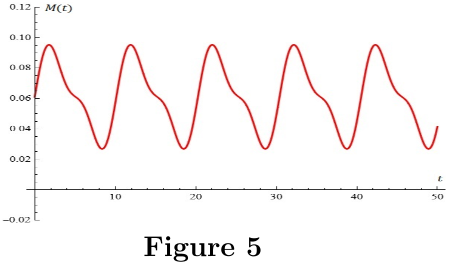

Figure 5

Proposition 2. If $N = 2$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, then the roots of Melnikov function $M(t_0)$ are given as solutions of the equation

$$M(t_0) = a + a_1\pi\omega csch(\pi\omega)sin(\omega t_0) + 2a_2\pi\omega csch(2\pi\omega)sin(2\omega t_0) = 0.$$

For example for $N = 2$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, $\omega = 0.38$, $a = 0.09$, $a_1 = 0.137$; $a_2 = 0.09$ the Melnikov function $M(t_0) = 0$ is depicted in fig. 4.

For $N = 2$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, $\omega = 0.62$, $a = 0.061$, $a_1 = 0.05$, $a_2 = 0.07$ the Melnikov function $M(t_0) = 0$ has no roots (fig. 5).

3.3. General case

The research conducted above gives us a reason to formulate the main result.

Proposition A. For some N and fixed $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, the roots of the Melnikov function $M(t_0)$ are given as solutions of the equation

$$M(t_0) = a + \pi\omega \sum_{i=1}^{N} i a_i \frac{sin(i\omega t_0)}{sinh(i\pi\omega)} = 0. \qquad (5)$$

For example for $N = 3$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$, $\omega = 0.28$, $a = 0.09$, $a_1 = 0.137$; $a_2 = 0.09$, $a_3 = 0.2$ the Melnikov function $M(t_0) = 0$ is depicted in fig. 6.

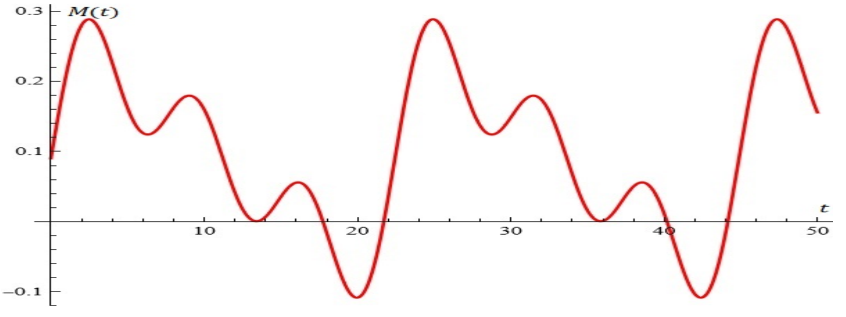

Figure 6

From the above examples, the reader can himself formulate the corresponding Melnikov criterion for the occurrence of chaos in the considered dynamic system for some N.

4. Some simulations

We will look at some interesting simulations on model (2):

Example 1. For given $N = 1$, $\omega = 0.1$, $a = 0.1$, $a_1 = 0.05$, $\epsilon = 0.03$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$ the simulations on the system (2) for $x_0 = 0.9$; $y_0 = 0.1$ are depicted on fig. 7.

Example 2. For given $N = 3$, $\omega = 0.1$, $a = 0.1$, $a_1 = 0.05$, $a_2 = 0.02$, $a_3 = 0.03$, $\epsilon = 0.03$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$ the simulations on the system (2) for $x_0 = 0.9$; $y_0 = 0.1$ are depicted on fig. 8.

Example 3. For given $N = 5$, $\omega = 0.1$, $a = 0.1$, $a_1 = 0.05$, $a_2 = 0.1$, $a_3 = 0.03$, $a_4 = 0.11$, $a_5 = 0.01$, $\epsilon = 0.03$, $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$ the simulations on the system (2) for $x_0 = 0.8$; $y_0 = 0.2$ are depicted on fig. 9.

Example 4. For given $N = 7$, $\omega = 0.95$, $a = 0.05$, $a_1 = 0.5$, $a_2 = 0.01$, $a_3 = 0.1$, $a_4 = 0.4$, $a_5 = 0.2$, $a_6 = 0.05$, $a_7 = 0.3$, $\epsilon = 0.07$, $\alpha = 1$, $\beta =$

2, $\gamma = -1$, $\delta = -2$ the simulations on the system (2) for $x_0 = 0.8$; $y_0 = 0.2$ are depicted on fig. 10.

With Examples 2-4, we demonstrate simulations of generated chaos in the new model.

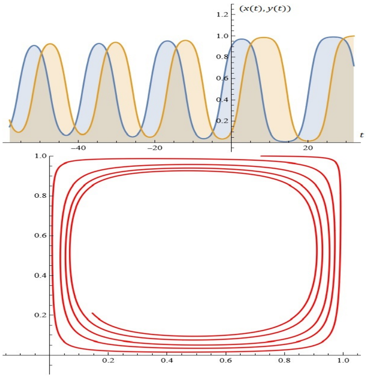

Figure 7. a) solutions of system (2) and phase space (Example 1)

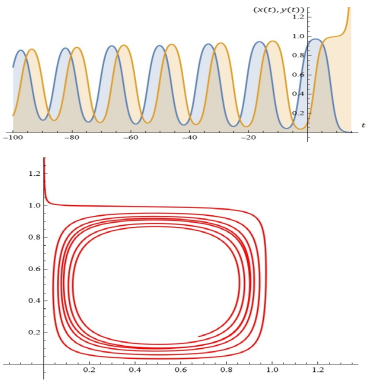

Figure 8. a) solutions of system (2) and phase space (Example 2)

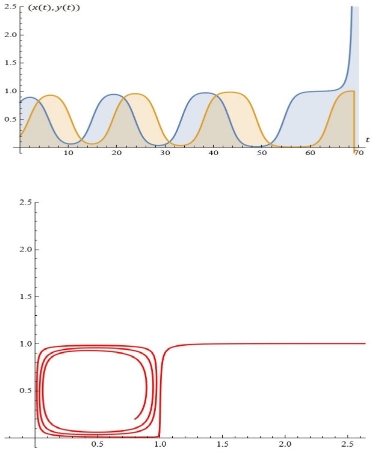

Figure 9. a) solutions of system (2) and phase space (Example 3)

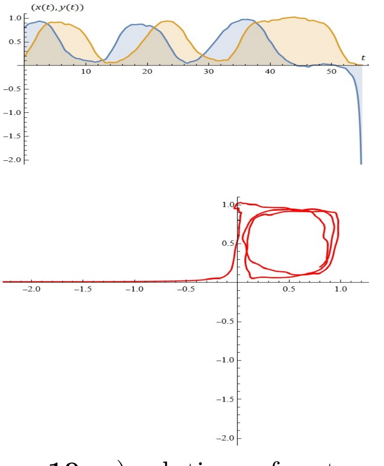

Figure 10. a) solutions of system (2) and phase space (Example 4)

5. A possible application of Melnikov functions in modelling and synthesis of radiating antenna patterns

Let us now focus on $M(t)$. In our previous publications, we discussed a possible application of the Melnikov functions in modeling and synthesis of radiation antenna diagrams. For more details see for example (Kyurkchiev & Andreev 2014), (Apostolov et al. 2018), (Kyurkchiev 2020), (Kyurkchiev et al. 2024).

We define the hypothetical normalized antenna factor as follows: $M^*(\theta) = \frac{1}{D}|M(K\cos\theta + k_1)|$ where θ is the azimuth angle; $K = kd$; $k = \frac{2\pi}{\lambda}$; λ is the wave length; d is the distance between emitters; k_1 is the phase difference.

Example 5. For fixed $N = 6$, $k = 8$, $k_1 = 0$, $\omega - 0.145$, $a = 0.004$, $a_1 = 0.137$, $a_2 = 0.09$, $a_3 = 0.2$, $a_4 = 0.3$, $a_5 = 0.3$, $a_6 = 0.2$ the Melnikov function and Melnikov antenna factor are depicted on fig. 11.

Example 6. For fixed $N = 10$, $k = 8$, $k_1 = 0$, $\omega - 0.145$, $a = 0.001$, $a_1 = 0.01$, $a_2 = 0.002$, $a_3 = 0.01$, $a_4 = 0.01$, $a_5 = 0.01$, $a_6 = 0.01$, $a_7 = 0.01$, $a_8 = 0.01$, $a_9 = 0.01$, $a_{10} = 0.2$ the Melnikov function and Melnikov antenna factor are depicted on fig. 12.

Example 7. For fixed $N = 12$, $k = 8$, $k_1 = 0$, $\omega - 0.145$, $a = 0.0001$, $a_1 = 0.01$, $a_2 = 0.002$, $a_3 = 0.01$, $a_4 = 0.01$, $a_5 = 0.01$, $a_6 = 0.01$, $a_7 = 0.01$, $a_8 = 0.01$, $a_9 = 0.01$, $a_{10} = 0.2$, $a_{11} = 0.1$, $a_{12} - 0.3$ the Melnikov function and Melnikov antenna factor are depicted on fig. 13.

Of course, this relatively new idea of justification and right to exist is subject to serious research by specialists working in this scientific direction.

The issue related to noise minimization (in decibels) also remains open.

6. Concluding Remarks

1. The discussions in this article are valid subject to the following limitations: $0 < \frac{\alpha}{\beta} < 1$, $0 < \frac{\gamma}{\delta} < 1$, $\alpha\gamma < 0$, and $\omega > 0$.

The existence of subharmonic periodic solutions of system (2) can be proved in the manner detailed in paper (Christie et al. 1995).

The system (1) ($\epsilon = 0$) has the Hamiltonian $H(x, y) = xy(1-x)(1-y) = h$. It is known (Christie et al. 1995) that the family of periodic orbit surrounding the centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ with assumptions $\alpha = 1$, $\beta = 2$, $\gamma = -1$, $\delta = -2$ is of the form

$$q_k(t) = (x_k(t), y_k(t)) = \left(\frac{1}{2} - \frac{k}{2} sn(\frac{t}{2}, k), \frac{1}{2} + \frac{k}{2} \frac{cn(\frac{t}{2}, k)}{dn(\frac{t}{2}, k)}\right)$$

where $k = \sqrt{1 - 16h}$, and $sn(u, k), cn(u, k), dn(u, k)$ are Jacobi elliptic functions with modulus k.

The orbit q_k has period $T(k) = 8K(k)$, $k \in (0, 1)$ and $K(k)$ is the complete elliptic integral of the first kind.

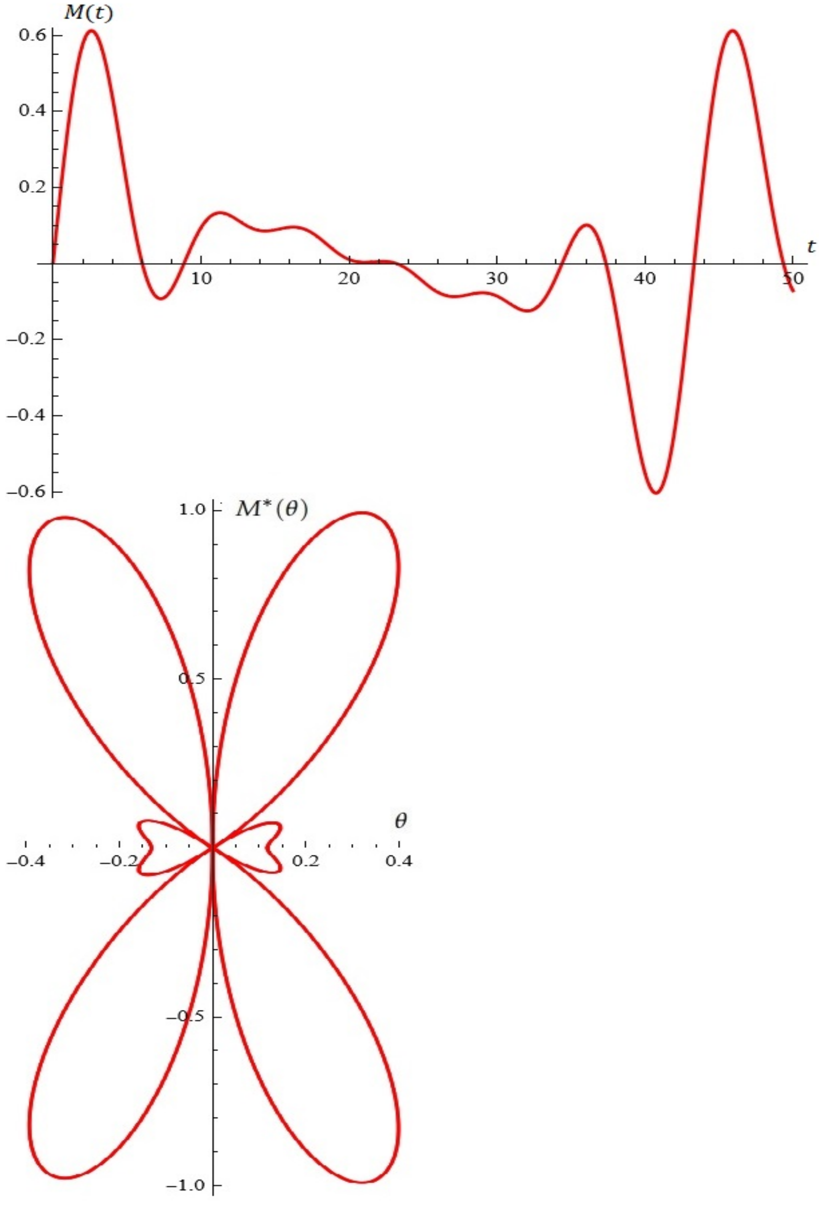

Figure 11. Melnikov function and Melnikov antenna factor (Example 5)

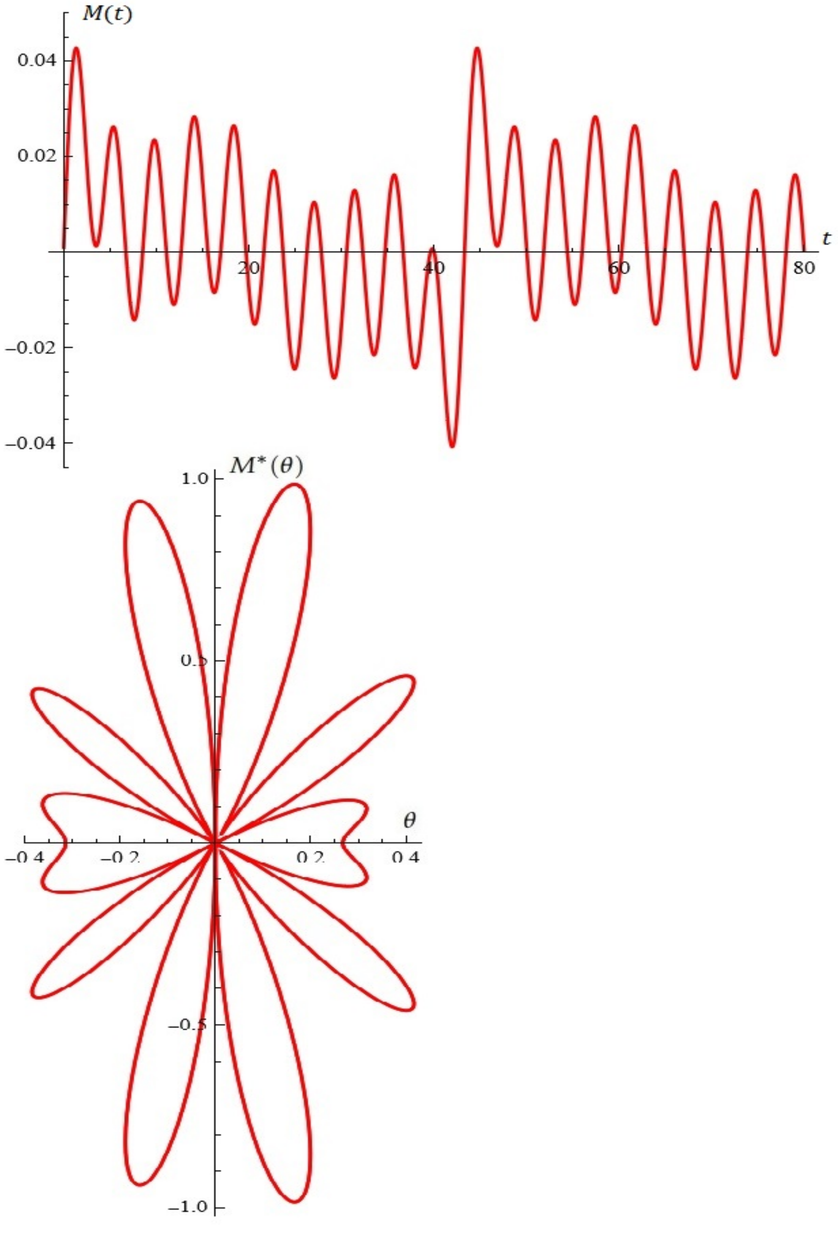

Figure 12. Melnikov function and Melnikov antenna factor (Example 6)

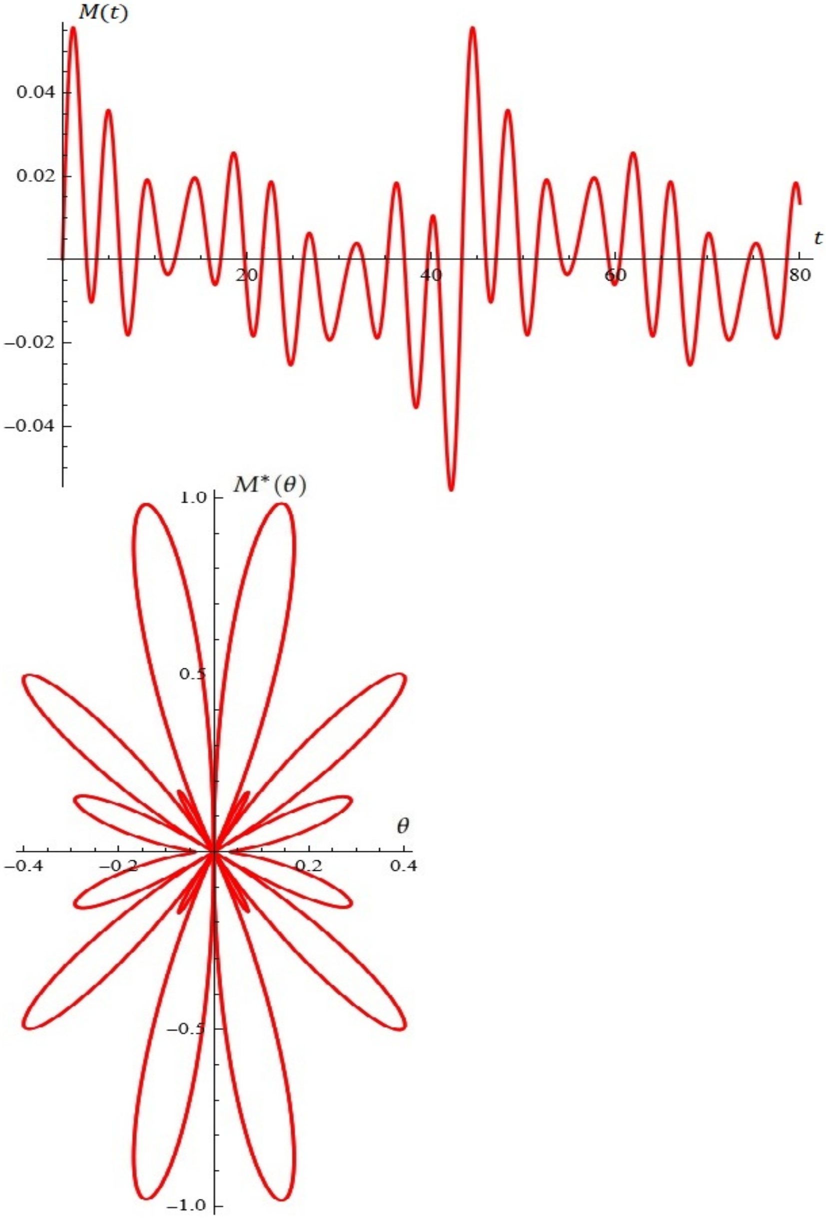

Figure 13. Melnikov function and Melnikov antenna factor (Example 7)

For relatively positive integers m and n the subharmonic Melnikov integral is defined as

$$M^{m/n}(t_0) = \int_{-\frac{mT}{2}}^{\frac{mT}{2}} f(q_k(t)) \wedge g(q_k(t), t + t_0)dt$$

and resonance condition is $T(k) = mT$.

For some details see (Christie et al. 1995).

We note that for our new model (2) the first subharmonic Melnikov function is:

$$M^{m/1}(t_0) = \\
= \int_{-\frac{mT}{2}}^{\frac{mT}{2}} \left(x_k(t)(1 - x_k(t))(1 - 2y_k(t)) \left(ay_k(t) + \sum_{j=1}^{N} a_j sin(j\omega(t + t_0)) \right) \right. \\
\left. -y_k(t)(1 - y_k(t))(-1 + 2x_k(t)) \left(ax_k(t) + \sum_{j=1}^{N} a_j sin(j\omega(t + t_0)) \right) \right) dt. \quad (6)$$

We leave the calculation of the integrals of (6) to the reader.

Similar theoretical studies and simulations can be done on a wide class of reaction-kinetic models described in the literature.

The proposed new extended model contains many free parameters (the coefficients a_i; $i = 1, 2, \ldots, N$), which makes it attractive for use in the fields of biological applications (Murray 2001), chemistry, sociology, lifetime analysis, reaction kinetics (Michaelis & Menten 1913), biostatistics, population dynamic, medical research etc.

2. For example consider the following new predator-prey dynamical system:

$$\begin{cases} \dfrac{dx}{dt} = x(f - by) + \epsilon \left(ax + \sum_{j=1}^{N} a_j \sin(j\omega t) \right) \\ \dfrac{dy}{dt} = y(dx - c) + \epsilon \left(ay + \sum_{j=1}^{N} a_j \sin(j\omega t) \right) \end{cases} \quad (7)$$

where $0 \le \epsilon < 1$, $a_i \ge 0$; $i = 1, 2, \ldots, N$ and N is integer.

The first integral ($\epsilon = 0$): $F(x, y) = dx + by - c \ln x - f \ln y$ is constant on solution curves $(x(t), y(t))$.

Using the function $F(x, y)$, one can furthermore show that all solution curves in the positive quadrant are closed, that is, all such solutions are periodic. For some details see (Prelle & Singer 1983).

The new class of extended models of the type (7) occurs in modeling of "predator-prey" systems can be studied in detail using the structural approach applied in this article for the model (2). For some models see (Kyurkchiev & Boyadjiev 2021), (Kyurkchiev et al. 2022), (Kyurkchiev et al. 2022a).

We will look at the following simulations on model (7)

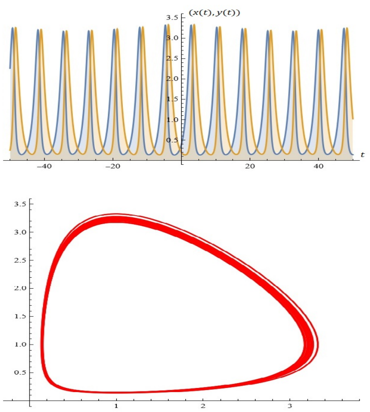

Figure 14. a) solutions of system (7) and phase space (Example 8)

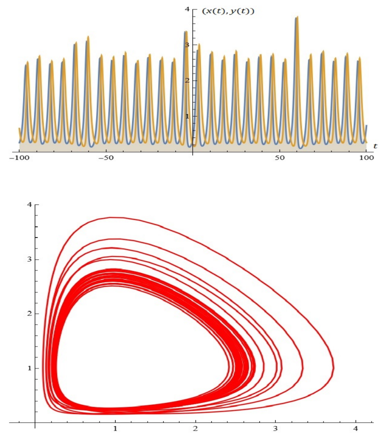

Figure 15. a) solutions of system (7) and phase space (Example 9)

Example 8. For given $N = 3$, $\omega = 0.1$, $a = 0.1$, $a_1 = 0.05$, $a_2 = 0.01$, $a_3 = 0.02$, $\epsilon = 0.03$, $f = b = 1$, $c = d = -1$ the simulations on the system (7) for $x_0 = 0.4$; $y_0 = 0.2$ are depicted on fig. 14.

Example 9. For given $N = 5$, $\omega = 0.1$, $a = 0.9$, $a_1 = 0.2$, $a_2 = 0.6$, $a_3 = 0.8$, $a_4 = 0.7$, $a_5 = 0.9$, $\epsilon = 0.03$, $f = b = 1$, $c = d = -1$ the simulations on the system (7) for $x_0 = 0.4$; $y_0 = 0.2$ are depicted on fig. 15.

3. We will note that the dynamics of the genetic model for the "battle of the sexes" by (Smith & Hofbauer 1987) can be studied with the methodology proposed in this article.

The derived results can be used as an integral part of a much more general application for scientific computing – for some details see (Golev et al. 2024), (Kyurkchiev et al. 2024), (Kyurkchiev et al. 2024a), (Kyurkchiev et al. 2024b), (Kyurkchiev & Zaevski 2023), (Kyurkchiev et al. 2023), (Kyurkchiev & Iliev 2022), (Kyurkchiev & Andreev 2014), (Apostolov et al. 2018), (Kyurkchiev 2020), (Kyurkchiev et al. 2024c), (Kyurkchiev et al. 2024d), (Kyurkchiev et al. 2024e), (Vasileva et al. 2024).

4. Nonstandard numerical methods connected to the investigation of the roots of equation $M(t_0) = 0$ can be found in (Proinov & Vasileva 2020), (Ivanov 2024).

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