Educational Technologies
Образователни технологии

A NEW PROOF OF THE FEUERBACH THEOREM

[1]Sava Grozdev, Hiroshi Okumura, Deko Dekov
[1] University of Finance, Business and Entrepreneurship – Sofia

Abstract. By using the Paskalev-Tchobanov distance formula, we give a new proof of the famous Feuerbach theorem.
Keywords: Paskalev-Tchobanov distance formula; Feuerbach theorem

The famous Feuerbach theorem states (See e.g. (Weisstein)):

Theorem 1. The nine-point circle of any triangle is tangent internally to the incircle and tangent externally to the three excircles.

Proof. Given triangle *ABC* with side-lengths $a = BC, b = CA, c = AB$. Denote by

– $s = \dfrac{a+b+c}{2}$ the semiperimeter of triangle *ABC*.
– Δ the area of triangle *ABC*.
– r the radius of the Incircle.
– R the radius of the Circumcircle.
– r_N the radius of the Nine-Point Circle.
– r_a the radius of the A-Excircle.
– r_b the radius of the B-Excircle.
– r_c the radius of the C-Excircle.

We use the following well-known formula e:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \quad r = \frac{\Delta}{s}. \quad R = \frac{abc}{4\Delta}, \quad R_N = \frac{R}{2}, \quad r_a = \frac{\Delta}{s-a},$$
$$r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}.$$

Denote

$$E = a^3 + b^3 + c^3 + 3abc - a^2b - ab^2 - a^2c - ac^2 - b^2c - bc^2,$$
$$E_1 = a^3 - b^3 - c^3 + 3abc + a^2b - ab^2 + a^2c - ac^2 + b^2c + bc^2,$$
$$E_2 = -a^3 + b^3 - c^3 + 3abc - a^2b + ab^2 + a^2c + ac^2 + b^2c - bc^2,$$
$$E_3 = -a^3 - b^3 + c^3 + 3abc + a^2b + ab^2 - a^2c + ac^2 - b^2c + bc^2.$$

Note that the above forms E_1, E_2, E_3 are used in (Grozdev, Okumura & Dekov, 2018). By using the above forms and formulae, we see easily, that

$$r_N - r = \frac{E}{8\Delta},\ \ r_N + r_a = \frac{E_1}{8\Delta},\ \ r_N + r_b = \frac{E_2}{8\Delta},\ \ r_N + r_c = \frac{E_3}{8\Delta}.$$

The None-Point Circle is tangent internally to the Incircle iff $NI = r_N - r$ and it is tangent externally to the A-Excircle iff $NJ_a = r_N + r_a$. Similar results hold for the B-Excircle and C-Excircle. The barycentric coordinates of the centers of the above circles are as follows (See e.g. (Grozdev & Dekov, 2016), (Weisstein)):

$$N = a^2b^2 + a^2c^2 + 2b^2c^2 - b^4 - c^4 ::$$
$$I = (a,b,c),\ \ J_a = (-a,b,c),\ \ J_b = (a,-b,c),\ \ J_c = (a,b,-c).$$

Now we can use the Paskalev-Tchobanov distance formula (See (Paskalev & Tchobanov, 1985), or formula (9) in (Grozdev & Dekov, 2016)). By using the Paskalev-Tchobanov distance formula we obtain $NI = \frac{E}{8\Delta}$. Hence $r_N - r = NI$. Similarly, we see that $r_N + r_a = NJ_a$, $r_N + r_b = NJ_b$ and $r_N + r_c = NJ_c$. This completes the proof.

REFERENCES

Grozdev, S. & Dekov, D. (2016). Barycentric Coordinates: Formula Sheet, *International Journal of Computer Discovered Mathematics*, vol.1, no 2, 75 – 82, available at the web.

Grozdev, S., Okumura H. & Dekov D. (2018) A Note on the Feuerbach triangle, *Mathematical Gazette*, vol.102, no.553, 135 – 138.

Paskalev, G. & Tchobanov I. (1985). *Remarkable points in triangle*. Sofia: Narodna Prosveta (in Bulgarian).

Weisstein, E. W. *MathWorld* – A Wolfram Web Resource, available at the web.

✉ **Prof. Sava Grozdev, DSc.**
Web of Science Researcher ID_AAG-4146-2020
ORCID 0000-0002-1748-7324
University of Finance, Business Entrepreneurship
1, Gusla St.
1618 Sofia, Bulgaria
E-mail: sava.grozdev@gmail.com

✉ **Prof. Dr. Hiroshi Okumura**
Maebashi Gunma, 371-0123
Japan
E-mail: hokmr@yandex.com

Dr. Deko Dekov, Assoc. Prof.