A NEW MEANING OF THE NOTION "EXPANSION OF A NUMBER"

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Abstract. In the present paper a new meaning of the notion "expansion of a number" is proposed by the authors. Some interesting problems are considered with corresponding answers and summarized results. The idea is by means of only one digit (used several times) and different mathematical symbols, signs and operations, to express (to expand) other digits. The research could be used by students and teachers to generate new examples with educational purposes or to find alternative variants of the problems in the paper.

Keywords: digit; mathematical sign; operation; expansion

Not rarely, there are problems in mathematical contests and quizzes connected with a clever use of mathematical procedures, operations, signs and symbols. Consider the following example:

Problem 1. Present the digit 9 using the digit 2 exactly five times in combination with mathematical symbols, signs and operations.

Solution: This is a simple problem and one of its solutions is

$$2.2.2 + 2 : 2 = 9\,.$$

In the present paper we call this procedure "expansion" and we use the phrase "expand the number", usually presenting the answer in the reverse order, i.e.

$$9 = 2.2.2 + 2 : 2\,.$$

Different modifications of this problem are interesting too:

– The digit 2 may be substituted by another one;
– The digit 9 may be substituted by another one;
– The number of digits in the expansion may be different.

Such problems usually help students, especially grass-roots, to attain proficiency in the proper use of mathematical symbols, signs and operations and also to learn how to think in a more rational way. Of course, there are symbols and operations that are present in the school curriculum at a later stage which can be the motivation for older and advanced students to solve such problems or at least to find different answers.

A summary of the above considerations can be formalized in the following:

Theorem 1. If n and m are digits and k is an integer greater than 1, there is at least one combination of mathematical symbols, signs and operations so that m may be expanded by using k times n, putting symbols and signs for operations in the right hand side of the equality (1), not obligatory between each two digits:

(1) $m = \underbrace{n\ n\ n \ldots n}_{k\ times}$.

Let us list all the signs, symbols and operations that may be used during the process:

"+", "–", ".", ":", "()", "$\sqrt{\ }$", "!", "!!", "[]", "][".

Some clarifications are needed:

– Only the square radical $\sqrt{\ }$ could be used. For other radicals new digits should be introduced.

– The factorial symbol "!" could be used together with a digit, i.e. $n! = 1.2...n$ $(0! = 1! = 1)$.

– The symbol "!!" is read "double factorial" and could be used after an even or an odd digit, for example: $6!! = 2.4.6 = 48$ and $7!! = 1.3.5.7 = 105$.

– If a is a real number, then $[a]$ denotes the integer part of a (sometimes $[a]$ is called *floor function* and is denoted by $\lfloor a \rfloor$), meaning the biggest integer which is less or equal to a, for example $[3,14] = 3$ and $[3] = 3$.

– If a is a real number, then $]a[$ denotes the smallest integer which is greater or equal to a (sometimes $]a[$ is called *ceiling function* and is denoted by $\lceil a \rceil$), or example $]3,14[= 4$. It is clear that $]3[= 3$ but if a is an integer, then we will use the previous function, i.e. $[3] = 3$.

Lemma 1. In terms of Theorem 1 when $k = 3$ and $n = 1$ there is at least one combination of symbols, signs and operations to expand a digit $m \in \{0, 1, 2, \ldots, 9\}$.

Proof:

1. If $m = 0$, then $0 = (1-1).1$;
2. If $m = 1$, then $1 = 1 + 1 - 1$;
3. If $m = 2$, then $2 = (1+1).1$;
4. If $m = 3$, then $3 = 1 + 1 + 1$;
5. If $m = 4$, then $4 = \left]\sqrt{11-1}\right[$;
6. If $m = 5$, we may apply the previous expansion: $5 = \left]\sqrt{(\left]\sqrt{11-1}\right[)!}\right[$.

Indeed $4!=24$, $4<\sqrt{24}<5$ and $5=]\sqrt{24}[$;

7. If $m=6$, then $6=(1+1+1)!$;

8. If $m=7$, from $(1+1+1)!=3!=6,\ 6!!=2.4.6=48$, $6<\sqrt{48}<7$ and $]\sqrt{48}[=7$, we receive $7=]\sqrt{((1+1+1)!)!!}[$;

9. If $m=8$ and $m=9$, we may apply the previous expansion:

$7!=5040$, $70<\sqrt{5040}<71$, $8<\sqrt{\sqrt{5040}}<9$, $8=[\ \sqrt{\sqrt{5040}}\ ]$ and $9=]\ \sqrt{\sqrt{5040}}\ [$. Hence $8=[\ \sqrt{\sqrt{(]\sqrt{((1+1+1)!)!!}[)!}}\ ]$ and $9=]\ \sqrt{\sqrt{(]\sqrt{((1+1+1)!)!!}[)!}}\ [$.

Lemma 2. There is at least one possible way to expand the digit 1 using any other digit once.

Proof:

$1=0!$, $1=1$, $1=[\sqrt{2}]$, $1=[\sqrt{3}]$, $1=[\sqrt{\sqrt{4}}]$, $1=[\sqrt{\sqrt{5}}]$, $1=[\sqrt{\sqrt{6}}]$, $1=[\sqrt{\sqrt{7}}]$, $1=[\sqrt{\sqrt{8}}]$, $1=[\sqrt{\sqrt{9}}]$.

If instead of 1 in Lemma 1 we use any expansion from Lemma 2, we obtain the following:

Lemma 3. In terms of Theorem 1 when $k=3$ and $n\in\{0,1,2,...,9\}$ there is at least one combination of symbols, signs and operations to expand any digit $m\in\{0,1,2,...,9\}$.

Note: Only once in the proof of Lemma 1 a sign is missing between two digits (in the case $m=4$) and we could have some doubt if for example $[\sqrt{3}][\sqrt{3}]$ is interpreted as 11. For this reason, we will suggest one more expansion in the case $m=4$, namely $4=\left]\sqrt{\sqrt{(]\sqrt{((1+1+1)!)!!}[)!!}}\right[$.

Indeed $(1+1+1)!=3!=6$, $6!!=48$, $6<\sqrt{48}<7$, $\left]\sqrt{48}\right[=7$, $7!!=1.3.5.7=105$, $3<\sqrt{\sqrt{105}}<4$ and $\left]\sqrt{\sqrt{105}}\right[=4$.

Lemma 4. In terms of Theorem 1 when $k=2$ and $n\in\{0,1,2,...,9\}$ there is at least one combination of symbols, signs and operations to expand any digit $m\in\{0,1,2,...,9\}$.

Proof: Firstly, we will mention that if we have two digits only, we cannot manage without “11”. Then we will show the expansion of any m by means of each n. (We skip $n=0$ because $1=0!$).

1. $n = 1$:

$0 = 1 - 1$, $1 = 1.1$, $2 = 1 + 1$, $3 = [\sqrt{11}]$, $4 =]\sqrt{11}[$, $5 =](\,]\sqrt{11}[\,)![$, $6 = ([\sqrt{11}])!$, $7 =]\sqrt{(([\sqrt{11}])!)!!}[$, $8 = \left[\sqrt{\sqrt{(\,]\sqrt{(([\sqrt{11}])!)!!}[\,)!}}\right]$, $9 =]\sqrt{\sqrt{(\,]\sqrt{(([\sqrt{11}])!)!!}[\,)!}}[$.

2. $n = 2$:

$0 = 2 - 2$, $1 = 2:2$, $2 = [(\sqrt{2} + \sqrt{2})]$, $3 =](\sqrt{2} + \sqrt{2})[$, $4 = 2 + 2$, $5 =]\sqrt{(2.2)!}[$, $6 = (\,](\sqrt{2} + \sqrt{2})[\,)!$, $7 =]\sqrt{((\,](\sqrt{2} + \sqrt{2})[\,)!)!!}[$, $8 = [\sqrt{\sqrt{7!}}]$ and $9 =]\sqrt{\sqrt{7!}}[$.

3. $n = 3$:

$0 = 3 - 3$, $1 = 3:3$, $2 = (3!):3$, $3 = [\sqrt{3} + \sqrt{3}]$, $4 =](\sqrt{3} + \sqrt{3})[$ and if we use the last expansion, then $5 =]\sqrt{4!}[$. Further, $6 = 3 + 3$ and from the last expansion we have $7 =]\sqrt{6!!}[$, $8 = [\sqrt{\sqrt{7!}}]$ and $9 =]\sqrt{\sqrt{7!}}[$.

4. $n = 4$:

$0 = 4 - 4$, $1 = 4:4$, $2 = [(\sqrt{\sqrt{4}} + \sqrt{\sqrt{4}})]$ and all formulas for $n = 2$ are re-calculated with $\sqrt{4}$. We have $n = 5: 0 = 5 - 5, 1 = 5:5, 2 = [\sqrt{\sqrt{5}}] + [\sqrt{\sqrt{5}}]$, $3 =](\sqrt{\sqrt{5}} + \sqrt{\sqrt{5}})[$, $4 = [\sqrt{5!:5}]$ and $5 =]\sqrt{5!:5}[$. From the expansion of 3 we have $6 = 3!$ and then consequently $7 =]\sqrt{6!!}[$, $8 = [\sqrt{\sqrt{7!}}]$ and $9 =]\sqrt{\sqrt{7!}}[$.

5. $n = 6$:

$2 = [\sqrt{6}]$ and we may use the results for “2”.

6. $n = 7$:

$2 = [\sqrt{7}]$ and we may use the results for “2”.

7. $n = 8$:

$2 = [\sqrt{8}]$ and we may use the results for “2”.

8. $n = 9$:

$3 = \sqrt{9}$ and we may use the results for “3”.

Thus, Lemma 4 is proved. We go back to the proof of the Theorem.

Case 1. $k=2l+1$ $(l=1,2,...)$. If $l=1$, see Lemma 3. If $l\geq 2$, then $\underbrace{n\ n\ n}_{3}\ \underbrace{n\ ...\ n}_{2l-2}$. For the first three digits we use Lemma 3 and for the rest we alternate the signs "+" and "−". Since the number of the remaining digits is even, we have

$$(n\ n\ n)+(n-n)+(n-n)+....+(n-n)=m+0=m.$$

Case2. $k=2l$ $(l=1,2,...)$. If $l=1$, see Lemma 4. If $l\geq 2$, then $\underbrace{n\ n}_{2}\ \underbrace{n\ n\ ...\ n}_{2l-2}$.

For the first two digits we use Lemma 4 and for the rest we alternate the signs "+" and "−". Since the number of the remaining digits is even, we have

$$(n\ n)+(n-n)+(n-n)+....+(n-n)=m+0=m.$$

Thus, the proof of the theorem is completed for $\forall k\geq 2$, $\forall n\in\{0,1,2,...,9\}$ and $\forall m\in\{0,1,2,...,9\}$.

In addition we will examine the case $k=1$. It is clear that using the afore mentioned symbols, signs and operations when $n=0$, only 0 and 1 should be expanded ($0=0$ and $1=0!$); when $n=1$ the expansion is possible only for 1 (1 = 1); when $n=2$ only 1 and 2 should be expanded ($1=\left[\sqrt{2}\right]$ and $2=2$).

Let us examine the situation, when $n\geq 3$. It is clear that $m=0$ may not be expanded.

1. $n=3$:

$1=[\sqrt{3}]$, $2=]\sqrt{3}[$, $3=3$, $4=[\sqrt{\sqrt{([\sqrt{\sqrt{(3!)!}}])!}}]$, $5=[\sqrt{\sqrt{(3!)!}}]$,

$6=3!$, $7=]\sqrt{(3!)!!}[$, $8=[\sqrt{\sqrt{7!}}]$, $9=]\sqrt{\sqrt{7!}}[$.

2. $n=4$:

$1=[\sqrt{\sqrt{4}}]$, $2=\sqrt{4}$, $3=]\sqrt{4!!}[$, $4=4$, $5=]\sqrt{4!}[$, $6=(\ ]\sqrt{4!!}[\ )!$, and consequently $7=]\sqrt{6!!}[$, $8=[\sqrt{\sqrt{7!}}]$, $9=]\sqrt{\sqrt{7!}}[$.

3. $n=5$:

$1=[\sqrt{\sqrt{5}}]$, $2=[\sqrt{5}]$, $3=]\sqrt{5}[$ and from the latter we may use the expansions for $n=3$. Similar ideas could be applied, when $n=6,7,8$, because $3=]\sqrt{6}[$, $3=]\sqrt{7}[$, $3=]\sqrt{8}[$ and when $n=9$, we have $3=\sqrt{9}$.

Summarizing, we obtain:

General theorem. If $k\geq 1$ is a natural number, $n\in\{0,1,2,...,9\}\equiv N$ and $m\in\{0,1,2,...,9\}\equiv M$, then:

1. When $k = 1$, for $\forall n \in N \setminus \{0,1,2\}$ at least one expansion for $\forall m \in M \setminus \{0\}$ is available.

2. When $k \geq 2$, for $\forall n \in N$ at least one expansion for $\forall m \in M$ is available.

Finally, we would like to make some conclusions and to propose some directions of future research on the topic.

1. Firstly, we should underline that the expansions given as examples are maybe not the most rational and easy ones. This gives opportunities to try finding some better options.

2. Secondly, we believe that the present paper will be quite helpful for students but also for a big number of teachers to generate problems of the type under consideration and to introduce them in the educational process. A possible example is the following: "In how many ways is it possible to expand the digit 7 using the digit 2 four times?".

3. Thirdly, the results in the paper are particular examples, examined by the authors. In terms of extending the topic some components may be changed:

– n may consist of different digits in the expansion of m ($m = n_1\ n_2\ ...\ n_k$);

– (and also n) may be a number (not only a digit), etc.

The authors strongly believe that by the present paper they have contributed to enrich the mathematical, logical and combinatorial abilities of the reader.

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ЕДНО НОВО ЗНАЧЕНИЕ НА ПОНЯТИЕТО „РАЗЛАГАНЕ НА ЧИСЛО“

Резюме. В настоящата статия авторите предлагат едно ново значение на понятието „разлагане на число“. Предложени са интересни задачи със съответни отговори и обобщени резултати. Идеята е с помощта на само една цифра (използвана няколко пъти) и на различни математически символи, знаци и операции да се представят (разложат) други цифри. Изследването може да се използва от учители и ученици за генериране на нови примери с образователни цели или да се намерят алтернативни варианти на задачите от статията.

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