https://doi.org/10.53656/math2025-4-1-gcs

Science in Education
Научно-методически статии

A GENERALIZATION OF THE COLLINEARITY OF THE STEREOGRAPHIC PROJECTION AROUND THE LEMOINE POINT

Vladislav Natchev
American College of Sofia (Bulgaria)

Abstract. Under stereographic projection, the projection point is collinear with the Lemoine points of the projection and the projected triangles or with the centers of their Apollonian circles (Natchev 2025). In the current paper, we generalize the discovered property of stereographic projection by proving that it applies to every point on the plane expressed in the barycentric form $(\delta a^2 : \varepsilon b^2 : \rho c^2)$, where $\delta, \varepsilon, \rho \in \mathbb{R}$. For a particular case, we investigate the points on the symmedians of the triangle and the tangents to the circumcircle at the vertices, where we derive collinearity of two more notable points of a triangle, namely the feet of the symmedians and the vertices of the tangential triangle. By setting the planimetric equivalent of the newly found facts, we connect the configurations they give rise to with Olympiad geometry.

Keywords: stereographic projection; collinearity; Olympiad geometry

1. Introduction

The first goal of the paper is to generalize the newly deduced facts for stereographic projection around the Lemoine point and the centers of the Apollonian circles of a triangle. The second goal of the article is to find applications of the generalization in discovering notable geometric configurations directly related to Olympiad mathematics.

In Section 2, we generalize the newly found property of stereographic projection for all points on the plane that can be described by a specific barycentric expression, and for the points on the symmedians of the triangle and the tangents to the circumcircle at the vertices, in particular. In Section 3, we investigate the applications of the generalization in Olympiad geometry. We formulate and solve synthetically five problems of an Olympiad-level complexity based on an examination of several special cases of the theorems introduced in Section 2.[1]

2. A Generalization of the Properties of Stereographic Projection around the Lemoine Point

It appears that the newly discovered properties around the Lemoine point and the centers of the Apollonian circles apply to all points in the plane. In Section 2, we discover the barycentric expression by which the points can be described to satisfy the property of collinearity. We also investigate two particular loci of points, namely the symmedians of a triangle and the tangents to the circumcircle at the vertices.

Theorem 2.1. Let under stereographic projection with a projection point O $\triangle ABC$ with sides a, b, c map to $\triangle A_1B_1C_1$ with sides a_1, b_1, c_1 (see fig. 1). Then point O, every point on the plane through $\triangle ABC$, expressed in the barycentric form $(\delta a^2 : \varepsilon b^2 : \rho c^2)$, and the point on the plane through $\triangle A_1B_1C_1$, expressed in the barycentric form $(\delta a_1^2 : \varepsilon b_1^2 : \rho c_1^2)$, are collinear, where $\delta, \varepsilon, \rho \in \mathbb{R}$.

Proof. If we denote the reference sphere and the projection plane by Σ and μ, correspondingly, then, by definition of stereographic projection, it follows that the plane μ is perpendicular to the diameter in Σ through O. Therefore, there exists a single sphere $i(O, r)$ with a center point O such that the plane μ is the radical plane of the spheres Σ and i. Therefore, under inversion Φ with an inversion sphere $i(O, r)$, $\Sigma \xrightarrow{\Phi} \mu$ (and so $A \xrightarrow{\Phi} A_1$, $B \xrightarrow{\Phi} B_1$, and $C \xrightarrow{\Phi} C_1$).

By applying the metric property of inversion (Johnson 1960, p. 48), $a_1 = \frac{r^2.a}{OC.OB}$, $b_1 = \frac{r^2.b}{OA.OC}$, and $c_1 = \frac{r^2.c}{OB.OA}$. Let us take a point P on the plane through $\triangle ABC$ with barycentric coordinates with respect to the triangle $(\delta a^2 : \varepsilon b^2 : \rho c^2)$, where $\delta, \varepsilon, \rho \in \mathbb{R}$.

It is clear that $\overrightarrow{OP} = \dfrac{\delta a^2.\overrightarrow{OA} + \varepsilon b^2.\overrightarrow{OB} + \rho c^2.\overrightarrow{OC}}{\delta a^2 + \varepsilon b^2 + \rho c^2}$, and so $\lambda_1 \overrightarrow{OP} = \delta a^2.\overrightarrow{OA} + \varepsilon b^2.\overrightarrow{OB} + \rho c^2.\overrightarrow{OC}$, where $\lambda_1 = \delta a^2 + \varepsilon b^2 + \rho c^2$. Let us denote the unit vectors with respect to $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$ by $\overrightarrow{OA_0}, \overrightarrow{OB_0}$, and $\overrightarrow{OC_0}$, correspondingly. Therefore, $\lambda_1 \overrightarrow{OP} = \delta a^2.OA.\overrightarrow{OA_0} + \varepsilon b^2.OB.\overrightarrow{OB_0} + \rho c^2.OC.\overrightarrow{OC_0}$.

Let us denote by P_1 the point on the plane μ with barycentric coordinates with respect to $\triangle A_1B_1C_1$ $(\delta a_1^2 : \varepsilon b_1^2 : \rho c_1^2)$. It is clear that $\overrightarrow{OP_1} = \dfrac{\delta a_1^2.\overrightarrow{OA_1} + \varepsilon b_1^2.\overrightarrow{OB_1} + \rho c_1^2.\overrightarrow{OC_1}}{\delta a_1^2 + \varepsilon b_1^2 + \rho c_1^2}$, and $\lambda_2 \overrightarrow{OP_1} = \delta a_1^2.\overrightarrow{OA_1} + \varepsilon b_1^2.\overrightarrow{OB_1} + \rho c_1^2.\overrightarrow{OC_1}$, where $\lambda_2 = \delta a_1^2 + \varepsilon b_1^2 + \rho c_1^2$. Since $OA.OA_1 = r^2$, then $\overrightarrow{OA_1} = OA_1.\overrightarrow{OA_0} = \dfrac{r^2.\overrightarrow{OA_0}}{OA}$. Analogously for $\overrightarrow{OB_1}$ and $\overrightarrow{OC_1}$. Therefore, $\lambda_2 \overrightarrow{OP_1} = \dfrac{\delta a_1^2.r^2.\overrightarrow{OA_0}}{OA} +$

$\dfrac{\varepsilon b_1^2.r^2.\overrightarrow{OB_0}}{OB} + \dfrac{\rho c_1^2.r^2.\overrightarrow{OC_0}}{OC}$. Now $\lambda_3\overrightarrow{OP_1} = \dfrac{\delta a_1^2.\overrightarrow{OA_0}}{OA} + \dfrac{\varepsilon b_1^2.\overrightarrow{OB_0}}{OB} + \dfrac{\rho c_1^2.\overrightarrow{OC_0}}{OC}$, where $\lambda_3 = \dfrac{\lambda_2}{r^2}$.

We observe that the vectors $\overrightarrow{OA_0}, \overrightarrow{OB_0}$, and $\overrightarrow{OC_0}$ are linearly independent in $\mathbb{R}^3$, and so we can set a coordinate system with a center point O and a basis $\overrightarrow{OA_0}, \overrightarrow{OB_0}$, and $\overrightarrow{OC_0}$. Therefore, $\lambda_1\overrightarrow{OP} = (\delta a^2.OA, \varepsilon b^2.OB, \rho c^2.OC)$ and $\lambda_3\overrightarrow{OP_1} = (\dfrac{\delta a_1^2}{OA}, \dfrac{\varepsilon b_1^2}{OB}, \dfrac{\rho c_1^2}{OC})$.

Let $\kappa = \dfrac{\frac{\delta a_1^2}{OA}}{\delta a^2.OA}$. Thus, $\kappa = \dfrac{\frac{\delta a_1^2}{OA}}{\delta a^2.OA} = \dfrac{\frac{\delta a^2.r^4}{OB^2.OC^2}}{\delta a^2.OA^2} = \dfrac{\frac{\frac{r^4.b^2}{OA^2.OC^2}.\delta a^2.OA^2}{b^2.OB^2}}{\delta a^2.OA^2} = \dfrac{\frac{b_1^2.\delta a^2.OA^2}{b^2.OB^2}}{\delta a^2.OA^2} = \dfrac{b_1^2}{b^2.OB^2} = \dfrac{\frac{\varepsilon b_1^2}{OB}}{\varepsilon b^2.OB}$. Also $\kappa = \dfrac{b_1^2}{b^2.OB^2} = \dfrac{\frac{c^2}{b^2}\cdot\frac{OC^2}{OB^2}.b_1^2}{\frac{c^2}{b^2}.b^2.OC^2} = \dfrac{\frac{\frac{r^4.c^2}{OA^2.OB^2}}{\frac{r^4.b^2}{OA^2.OC^2}}.b_1^2}{\frac{c^2}{b^2}.b^2.OC^2} = \dfrac{\frac{c_1^2}{b_1^2}.b_1^2}{\frac{c^2}{b^2}.b^2.OC^2} = \dfrac{c_1^2}{c^2.OC^2} = \dfrac{\frac{\rho c_1^2}{OC}}{\rho c^2.OC}$. Therefore, $\kappa.\lambda_1.\overrightarrow{OP} = \lambda_3.\overrightarrow{OP_1}$, and so $\dfrac{\kappa\lambda_1}{\lambda_3}.\overrightarrow{OP} = \overrightarrow{OP_1}$. Consequently, the points O, P, and P_1 are collinear. □

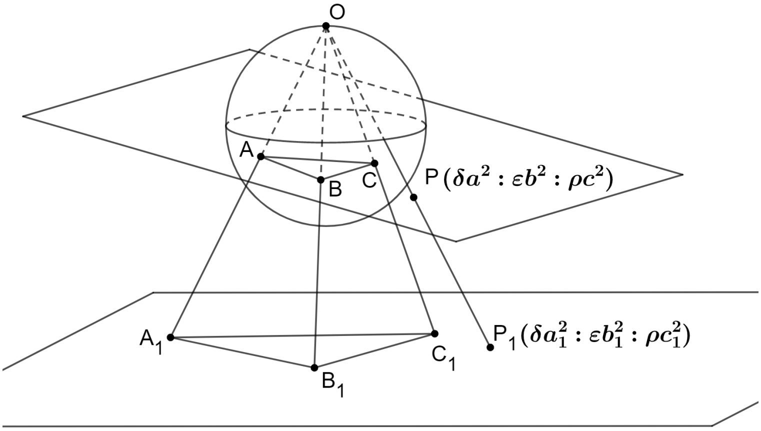

Figure 1

Remark. Since every point on the plane through $\triangle ABC$ can be expressed in the barycentric form $(\delta a^2 : \varepsilon b^2 : \rho c^2)$, then the theorem applies for all points on the plane.

Now we will reveal two direct results from the theorem, investigating two cases for the constants' values. If we set $\epsilon = \rho = 1$, then, taking into account that the barycentric equation for the A-symmedian of $\triangle ABC$ is $b^2 z - c^2 y = 0$, we get the following result.

Theorem 2.2. Let under stereographic projection with a projection point O $\triangle ABC$ with sides a, b, c map to $\triangle A_1B_1C_1$ with sides a_1, b_1, c_1 (see fig. 2). Then point O, every point on the A-symmedian for $\triangle ABC$, expressed in the barycentric form $(\delta a^2 : b^2 : c^2)$, and the point on the A_1-symmedian for $\triangle A_1B_1C_1$, expressed in the barycentric form $(\delta a_1^2 : b_1^2 : c_1^2)$, are collinear, where $\delta \in \mathbb{R}$.[2]

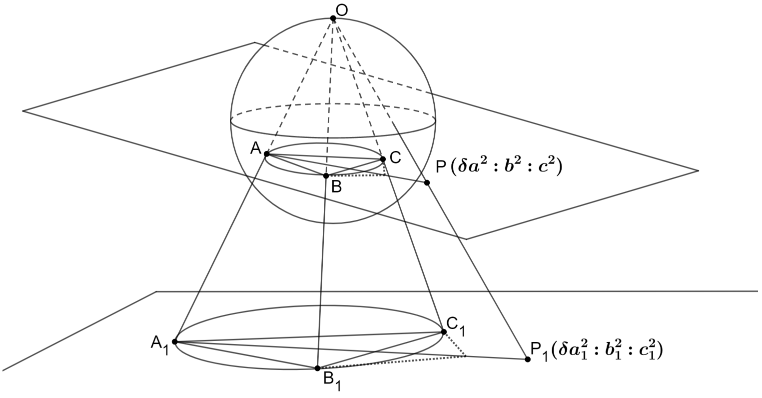

Figure 2

Since the barycentric equation for the A-tangent line, (Schindler & Chen 2012, p. 14), is $b^2 z + c^2 y = 0$, then by setting $\epsilon = 1$ and $\rho = -1$, we get the following fact as well.

Theorem 2.3. Let under stereographic projection with a projection point O $\triangle ABC$ with sides a, b, c map to $\triangle A_1B_1C_1$ with sides a_1, b_1, c_1 (see fig. 3). Then point O, every point on the A-tangent to the circumcircle of $\triangle ABC$, expressed in the barycentric form $(\delta a^2 : b^2 : -c^2)$, and the point on the A_1-tangent for $\triangle A_1B_1C_1$, expressed in the barycentric form $(\delta a_1^2 : b_1^2 : -c_1^2)$, are collinear, where $\delta \in \mathbb{R}$.[2]

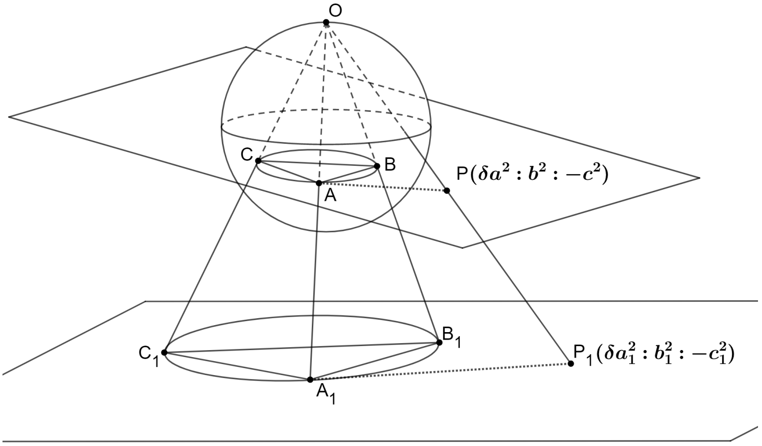

Figure 3

3. Applications in Olympiad Geometry

The deduced generalization of collinearity between the center of stereographic projection and given points of the projection and the projected triangles finds application in Olympiad geometry. For this purpose, we reformulate the statements in Theorems 2.2. and 2.3. by the following way: "Let us take the intersecting circles k_1, k_2, and k_3 in the 3D space such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Then point O, the point with barycentric coordinates $(\delta a^2 : b^2 : \pm c^2)$ with respect to $\triangle ABC$, and the point with barycentric coordinates $(\delta d^2 : e^2 : \pm f^2)$ with respect to $\triangle DEF$ are collinear, where $\delta \in \mathbb{R}$, and a, b, c and d, e, f are the sides of the two triangles." Point O lies on the radical line of the three circles, and so it will have the same power with respect to them, S. Therefore, we can take an inversion with an inversion sphere with a center point O and a radius $\sqrt{S}$, at which $\triangle ABC$ will map to $\triangle DEF$. It is clear that there exists a stereographic projection with a projection point O, a reference sphere passing through the points A, B, C, and O, and a projection plane passing through the points D, E, and F. As a result, the restatement is valid. As a consequence, for the particular case when the three circles lie on a plane, we get a set of Olympiad-level planimetric configurations.

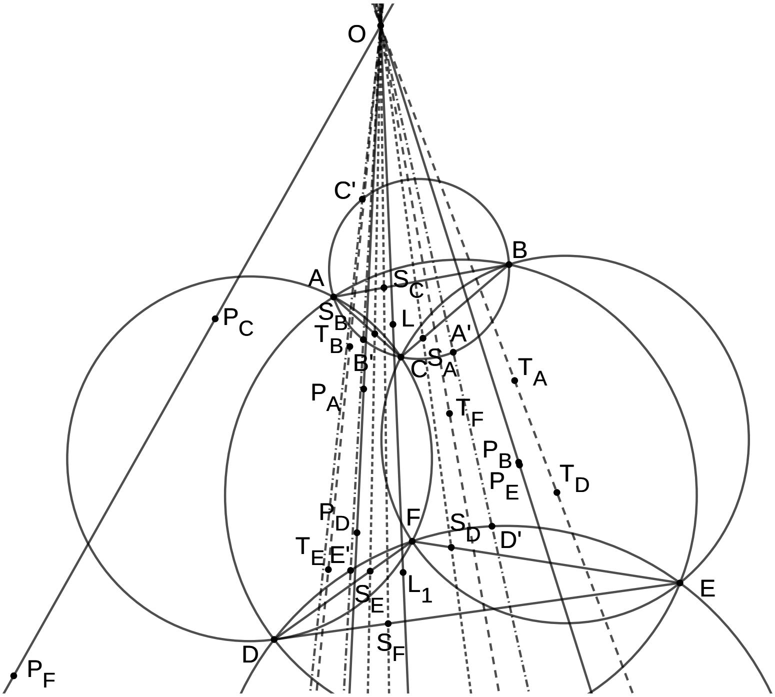

Figure 4

We examine the following special cases for the value of the parameter from Theorem 2.2., in which a given notable point passes from one triangle to another:

- $\delta = 1$ – Lemoine point (synthetically proven in (Natchev 2025)) (see fig. 4: the line O, L, L_1)
- $\delta = 0$ – foot of symmedian (see fig. 4: the line O, S_A, S_D with respect to A and D)
- $\delta = -\frac{1}{2}$ – intersection point of the symmedian with the circumcircle of the triangle (synthetically proven in (Natchev 2025)) (see fig. 4: the line O, A', D' with respect to A and D) (*Note:* the quadrilaterals $ABA'C$ and $DED'F$ are harmonic)
- $\delta = -1$ – vertex of the tangential triangle (see fig. 4: the line O, T_A, T_D with respect to A and D)

As well as for the value of the parameter from Theorem 2.3.:

- $\delta = 1$ – vertex of the tangential triangle
- $\delta = 0$ – center of an Apollonian circle (synthetically proven in (Natchev 2025)) (see fig. 4: the line O, P_A, P_D with respect to A and D)

Based on the discovered facts, we formulate five Olympiad problems and solve them synthetically. The collinearity of the Lemoine points, the intersection points of the symmedians with the circumcircles, and the centers of the Apollonian circles of the projection and the projected triangles is proven in (Natchev 2025) using inversion, polar reciprocation, and properties of symmedians and Apollonian circles.

Problem 3.1. Let us take the intersecting circles k_1, k_2, and k_3 in the plane such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the centers of the Apollonian circles for $\triangle ABC$, and the corresponding points for $\triangle DEF$ are collinear.

Problem 3.2. Let us take the intersecting circles k_1, k_2, and k_3 in the plane such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the intersection points of the symmedians with the circumcircle of $\triangle ABC$, and the corresponding points for $\triangle DEF$ are collinear.

Problem 3.3. Let us take the intersecting circles k_1, k_2, and k_3 in the plane such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the Lemoine point for $\triangle ABC$, and the Lemoine point for $\triangle DEF$ are collinear.

Problem 3.4. Let us take the intersecting circles k_1, k_2, and k_3 in the plane such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the feet of the symmedians for $\triangle ABC$, and the corresponding points for $\triangle DEF$ are collinear.

Proof. We will prove the statement with respect to the vertices A and D. Let us denote the feet of the symmedians by S_A and S_D, correspondingly (see fig. 4). From the cyclic quadrilateral $FEBC$, it follows that $\triangle OCB \sim \triangle OEF$. Thus, by Steiner isogonality (ratio) theorem, (Rong 2021, p. 4), we know that the points O, S_A, and S_D are collinear if and only if $\frac{CS_A}{S_AB} \cdot \frac{ES_D}{S_DF} = \frac{OE^2}{OF^2}$. From the similarity we have that $\frac{OC}{OB} = \frac{OE}{OF}$. But from the cyclic quadrilateral $DFCA$, it follows that $\triangle OAC \sim \triangle OFD$, from where $\frac{AC}{DF} = \frac{OC}{OD}$, and so $AC^2 = \frac{DF^2.OC^2}{OD^2}$. From the cyclic quadrilateral $ABED$, it follows that $\triangle OAB \sim \triangle OED$, from where $\frac{AB}{DE} = \frac{OB}{OD}$, and so

$AB^2 = \frac{DE^2.OB^2}{OD^2}$. From the symmedian property, we derive $\frac{CS_A}{S_AB} = \frac{AC^2}{AB^2}$ and $\frac{FS_D}{S_DE} = \frac{DF^2}{DE^2}$. Therefore, we have that $\frac{CS_A}{S_AB}.\frac{ES_D}{S_DF} = \frac{AC^2}{AB^2}.\frac{ES_D}{S_DF} = \frac{\frac{DF^2.OC^2}{OD^2}}{\frac{DE^2.OB^2}{OD^2}}.\frac{ES_D}{S_DF} = \frac{DF^2}{DE^2}.\frac{OC^2}{OB^2}.\frac{ES_D}{S_DF} = \frac{FS_D}{S_DE}.\frac{OE^2}{OF^2}.\frac{ES_D}{S_DF} = \frac{OE^2}{OF^2}$ that we wanted to prove. □

Problem 3.5. Let us take the intersecting circles k_1, k_2, and k_3 in the plane such that $k_1 \cap k_2 = \{A, D\}, k_2 \cap k_3 = \{B, E\}$, and $k_1 \cap k_3 = \{C, F\}$. Let $AD \cap BE \cap CF = O$. Prove that point O, the vertices of the tangential triangle for $\triangle ABC$, and the corresponding points for $\triangle DEF$ are collinear.

Proof. We will prove the condition with respect to the vertices A and D. Let us denote the vertices of the tangential triangles by T_A and T_D, correspondingly (see fig. 4). If we denote the other notable points analogously and the Lemoine points by L and L_1, we observe that $(A, S_A; L, T_A) \overset{C}{=} (AC \cap T_CT_A, B; T_C, T_A) = -1$ since $T_CC \cap T_AA \cap T_BB = L$. By Problems 3.3. and 3.4., we have that the points O, A, D; O, L, L_1; O, S_A, S_D are collinear. Let us construct the line through the points D, L_1, and S_D, and let us intersect it with the line OT_A at point T'_D. Therefore, $-1 = (A, S_A; L, T_A) \overset{O}{=} (D, S_D; L_1, T'_D)$, but there exists a single point T'_D from the line DL_1 such that the points D, L_1, S_D, and T'_D are in harmonic division in this order, and we know that T_D is such. Consequently, $T'_D \equiv T_D$, which we wanted to prove. □

4. Conclusion

We expand the newly discovered properties of stereographic projection around the Lemoine point in (Natchev 2025) to every point on the plane that can be expressed in the barycentric form $(\delta a^2 : \varepsilon b^2 : \rho c^2)$. Using inversion and vectors, we prove the desired collinearity of corresponding points between the projection and the projected triangles.

We draw a parallel between the proven generalization and Olympiad geometry. We reformulate the two theorems from Section 2 related to the points on the symmedians and the tangents to the circumcircle at the vertices so that we can evaluate the special case when the construction is planimetric. Then we examine several particular cases for the value of the constant from Theorems 2.2. and 2.3., which leads to the proof of collinearity of two additional notable points of a triangle under stereographic projection as well as to an analytical confirmation of the derived results in (Natchev 2025). Moreover, we formulate the newly found planimetric configurations as Olympiad problems and solve them synthetically using Olympiad methods.

Therefore, in this paper, we managed to: generalize the newly discovered properties of stereographic projection for all points (Theorem 2.1.), for the symmedians (Theorem 2.2.), and for the tangents to the circumcircle at the vertices of the triangle (Theorem 2.3.), formulate Olympiad problems derived from this generalization (Problems 3.1., 3.2., 3.3., 3.4., and 3.5.), and solve them synthetically.

In conclusion, we provide a generalization of the newly found facts around stereographic projection in (Natchev 2025), and as a consequence, we formulate five Olympiad problems. The discovered constructions would be significant for the development of Olympiad geometry and for encouraging further exploration of the topic.

NOTES

1. All statements in this paper refer to the three-dimensional Euclidean space unless otherwise stated.
2. Analogously, the theorem applies to the other vertices.

REFERENCES

JOHNSON, R., 1960. Coaxal Circles and Inversion. *Advanced Euclidean Geometry.* Dover, p. 48.

NATCHEV, V., 2025. Apollonian Sphere and Properties of Stereographic Projection around the Lemoine Point. *Mathematics and Informatics.* **68**(1). Available from: https://doi.org/10.53656/math2025-1-3-apo

RONG, V., 2021. Ratio Chasing. *Victor Rong Math Olympiad Handouts*, p. 4.

SCHINDLER, M. & CHEN, E., 2012. Barycentric Coordinates in Olympiad Geometry. *Evan Chen Olympiad Articles*, p. 14.

✉ **Vladislav Natchev**
ORCID iD: 0009-0008-9568-413X
American College of Sofia
Sofia, Bulgaria
E-mail: `v.natchev2007@gmail.com`