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SOME MATH QUIZZES FOR PRESENTING A NUMBER BY USING EQUAL DIGITS AND MATH OPERATIONS

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Abstract. This paper gives a formalization of a problem for presenting a number by using particular set of equal digits and a range of math operators. Based on the formal definition, an example for presenting 7 in two ways by using 4 times the digit 2 is given. The first option is without any restrictions and the objective is to have the maximum possible number of solutions. For the second option there are some restrictions concerning the type and number of operations. The aim here is to achieve a result with minimum deviation from the constraints. Both options can be used as a type of math quiz for improving the cognitive way of thinking.
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Math Quizzes can be determined as a specific type of problems with a gaming element. For their solutions we need more creativity and quickness of mind than using some specific erudition. They can be fun and they can be used for training smart and logical way of thinking.

One type of such quizzes is the problem for presenting a number by using some digits and math operations. Such an example is the problem for "the four fours". The objective in this problem is to find an algorithm (or algorithms) to determine the longest possible series of consequent natural numbers by using four times the digit 4 and applying unspecified number of times the basic math notations (Ball, 1914: 14). Basic math notations the author cited interprets as integers and fractions, factorial and subfactorial[1)], symbols, square root, addition, subtraction, multiplication, division, brackets used a finite number of times. In some sources (Bourke)[2)] the basic notations include the symbols %, $\sum$ etc.

A general type of problems for presenting one-digit integers by using a determined number of equal digits is described from Nikolaev, Milkova and Miryanov (2018). The authors prove that every one-digit integer can be presented (expanded) by using determined number of equal digits and some math operations, given in advance. The results enable the composition of math quizzes. The worthiness of this approach is the restriction over the used math operations and the latter are only those that are included in the basic math education or some others which are quite easy to be defined (for example double factorial or approximation).

Using the ideas from the last research above, we will try to give a particular example. We can classify the notations, treated in the source cited, in the following sets:

– $T_1 = \{+ ; - ; \cdot ; /\}$ – arithmetic operations addition, subtraction, multiplication and division;

– $T_2 = \{\wedge ; \sqrt{\ }\}$ – power and root operations. If it is a square root, no digit is deemed to be used;

– $T_3 = \{\lfloor\ \rfloor ; \lceil\ \rceil\}$ – approximation procedures (floor and ceiling functions);

– $T_4 = \{\,! ; !!\,\}$ – factorial and double factorial operations, where

$0! = 1$ and $n! = n \cdot (n-1)!$ for all $n \geq 1$;

$0!! = 1!! = 1$ and $n!! = n \cdot (n-2)!!$ for all $n \geq 2$;

– $T_5 = \{(\ )\}$ – notation for brackets, i.e. setting priority of procedures.

This classification shows only the different level of erudition, expected from a person, solving such type of math quizzes.

The glamour of math quizzes is hidden in the fact that their solutions are not based on any algorithm, known in advance, but need non-standard approach for each separate problem. "Unfortunately," it has been already proved (Nikolaev et al., 2018), that by means of series of roots and approximations, all the one-digit numbers can be easily presented using two or more equal digits. For that reason and for composing an interesting and difficult problem, we will consider some additional restrictions. Such, for example, can be the obligatory use of a particular operation given number of times, or number of times in a given range. All possible options are given by the Cartesian product

(1) $P = \{T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5\} \times N_0$, where:

N_0 – set of natural numbers plus 0.

If *Op* is an operation from the sets T_i and $s < t$ are natural numbers, then the number of use of the operations may be given by the notations:

(2) *Op* $\langle 0 \rangle$ – the operation is forbidden;

(3) *Op* $\langle s \rangle$ – the operation is used exactly s times;

(4) *Op* $\langle s \ldots t \rangle$ – the operation is used from s to t times;

(5) *Op* $\langle s \ldots \rangle$ – the operation is used at least s times;

(6) *Op* $\langle \ldots \rangle$ or *Op* – the operation is used (or not) without any restrictions.

Then every math quiz from the aforementioned type can be formalized by the ordered set of four elements

(7) $Q = \{m, n, k, S\}$, where:

$m \in \{0,1,2,3,4,5,6,7,8,9\}$ is a one-digit number;

$n \in \{0,1,2,3,4,5,6,7,8,9\}$ is a digit in the decimal number system;

$k \in N$ is a natural number;

$S \subset P$.

The formalization (7) gives the information that the one-digit number m should be presented by using exactly k times the digit n and operations from the set P. For the sake of convenience, the set S is given by the notations (2)-(6).

As an example, let us take the following problem (Nikolaev et al. 2018):

Example 1. In how many different ways can 7 be expanded (presented) by using 4 times the digit 2?

We can afford to add "Without calculator or computer".

Using the formal definition (7) and the notations (2)-(6), we have:

$Q = \{7,2,4,S\}$, where:

$S = \{+\,;-\,;\cdot\,;/\,;\wedge;\sqrt{\ }\,;[\ ];]\ [;!;!!;(\ )\}$.

With such formalization there are no restrictions for the number of the operations used. We will show that the options for solutions are really various.

It has been proved in Lemmas 1 – 4 (Nikolaev et al. 2018), that all the one-digit numbers can be presented by using equal digits 2. If 2 is used once, then only $1 = \left[\sqrt{2}\right]$ and $2 = 2$ can be presented. If 2 is used from two to four times, then all the one-digit numbers can be presented.

On the other hand, 7 can be presented (expanded) by using not more than 4 elements in the following order:

1) 7=7;
2) 7=6+1;
3) 7=5+2;
4) 7=5+1+1;
5) 7=4+3;
6) 7=4+2+1;
7) 7=4+1+1+1;
8) 7=3+3+1;
9) 7=3+2+2;
10) 7=3+2+1+1;

11) 7=2+2+2+1.

From the aforementioned 11 expansions only 1), 2), 3), 4), 5), 6), 9) и 11) can be used as a base for the solution of the example. The expansion 7) would need at least 5 times the digit 2 (twice for the constant 4 and once more for each constant 1). For similar reasons the expansions 8) and 10) cannot be used.

Let us consider the expansion 1) which is not a real "expansion" in its meaning. Possible options are (without any claims for being comprehensive):

a) the idea from (Nikolaev et al., 2018):

since $\left]\sqrt{\left(\left(\left]\left(\sqrt{2}+\sqrt{2}\right)\right[\right)!\right)!!}\right[=7$, then obviously

$\left]\sqrt{\left(\left(\left]\left(\sqrt{2}+\sqrt{2}\right)\right[\right)!\right)!!}\right[+2-2=7$ and $\left]\sqrt{\left(\left(\left]\left(\sqrt{2}+\sqrt{2}\right)\right[\right)!\right)!!}\right[\cdot 2/2=7$.

b) by using multi-digit numbers with the digit 2:

Since

$6^4=1296<2222<2401=7^4$, then $\left]\sqrt{\sqrt{2222}}\right[=7$.

Since

(8) $14^2=196<222<225=15^2$, then $\left[\frac{\sqrt{222}}{2}\right]=7$.

в) by using factorials:

$$\left]\sqrt{(2+2)!+(2+2)!}\right[=\left]\sqrt{24+24}\right[=\left]\sqrt{48}\right[=7 \quad .$$

$$\left]\sqrt{(2+2+\sqrt{2+2})!!}\right[=\left]\sqrt{6!!}\right[=\left]\sqrt{48}\right[=7 \quad .$$

Options for expansion 2) are for example:

$$(2+2/2)!+\left[\sqrt{2}\right]=3!+1=6+1=7\ ;$$

$$\left(2+\left[\sqrt{2}\right]\right)!+2/2=3!+1=7\ ;$$

$$\left(\left]\left(\sqrt{2}+\sqrt{2}\right)\right[\right)!+2/2=3!+1=7\ ;$$

$$\left]\left(2\cdot 2\cdot\sqrt{2}\right)\right[+\left[\sqrt{2}\right]=6+1=7 \quad \text{etc.}$$

Options for expansion 3) are for example:

$$\left[\left(2\cdot 2\cdot \sqrt{2}\right)\right]+2=5+2=7\ ;$$

$$\left]\sqrt{(2\cdot 2)!}\right[+\sqrt{2\cdot 2}=\left]\sqrt{24}\right[+\sqrt{4}=5+2=7\ .$$

Options for expansion 4) are for example:

$$\left]\sqrt{(2\cdot 2)!}\right[+\left[\sqrt{2}\right]+\left[\sqrt{2}\right]=\left]\sqrt{24}\right[+1+1=5+1+1=7\ ;$$

$$\left]\sqrt{(2+2)!}\right[+\left[\sqrt{2}\right]+\left[\sqrt{2}\right]=\left]\sqrt{24}\right[+1+1=5+1+1=7\ ;$$

$$\left]\sqrt{(2^2)!}\right[+\left[\sqrt{2}\right]+\left[\sqrt{2}\right]=\left]\sqrt{24}\right[+1+1=5+1+1=7\ .$$

Options for expansion 5) are for example:

$$2\cdot 2+\left]2\sqrt{2}\right[=4+3=7\ ;$$

$$2^2+\left]2\sqrt{2}\right[=4+3=7\ .$$

Options for expansion 6) are for example:

$$2\cdot 2+2+\left[\sqrt{2}\right]=4+2+1=7\ ;$$

$$2^2+2+\left[\sqrt{2}\right]=4+2+1=7\ .$$

Options for expansion 9) are for example:

$$\left]2\cdot\sqrt{2}\right[+2+2=3+2+2=7\ ;$$

$$\left]2^{\sqrt{2}}\right[+2+2=3+2+2=7\ .$$

Options for expansion 11) are for example:

(9) $2+2+2+\left[\sqrt{2}\right]=2+2+2+1=7$;

$$\left]\sqrt{2}\right[+\left]\sqrt{2}\right[+\left]\sqrt{2}\right[+\left[\sqrt{2}\right]=2+2+2+1=7\ .$$

The number of options for almost all expansions can be increased “artificially” considering the obviously correct equalities:

$\underbrace{\left[\sqrt{\sqrt{\ldots\sqrt{2}}}\right]}_{n}=\left[\sqrt{2}\right]=1$ and $\underbrace{\left]\sqrt{\sqrt{\ldots\sqrt{2}}}\right[}_{n}=\left]\sqrt{2}\right[=2$ for every natural n.

Additional options can be generated by applying $7 = 8 - 1$, $7 = 9 - 2$ etc. For example, for $7 = 8 - 1$:

(10) $2 \cdot 2 \cdot 2 - \left[\sqrt{2}\right] = 8 - 1 = 7$;

$2 \cdot 2^2 - \left[\sqrt{2}\right] = 8 - 1 = 7$,

or a well-known elegant solution without any approximation used:

(11) $(2 \cdot 2)!! - 2/2 = 4!! - 1 = 8 - 1 = 7$.

By such means some other solutions may be composed.

Let us examine the expression:

$$A = \left(\underbrace{\sqrt{\ldots\sqrt{2}}}_{v_1} + \underbrace{\sqrt{\ldots\sqrt{2}}}_{v_2} + \underbrace{\sqrt{\ldots\sqrt{2}}}_{v_3} \right) \cdot 2,$$

where for the elements of the vector $V = (v_1, v_2, v_3)$ we assume that $v_1 \le v_2 \le v_3$, i.e. the order is lexicographic. It is obvious that for each V the inequality $A > 6$ is correct. It can be proved that for:

$V = (1,1,2) \Rightarrow A > 8$;

$V = (1,1,3) \Rightarrow 7 < A < 8$.

Indeed, if we assume that:

$\left(\sqrt{2} + \sqrt{2} + \sqrt{\sqrt{2}}\right) \cdot 2 > 8$, after some consecutive transformations we have:

$\left(2\sqrt{2} + \sqrt{\sqrt{2}}\right) > 4$,

$\sqrt{\sqrt{2}} > 4 - 2\sqrt{2}$,

$\left(\sqrt{\sqrt{2}}\right)^2 > \left(4 - 2\sqrt{2}\right)^2$,

$\sqrt{2} > 16 - 16\sqrt{2} + 8$,

$17\sqrt{2} > 24$,

$17^2 \cdot 2 > 24^2$,

$588 > 576$, which is correct.

Let us assume that

$7<\left(\sqrt{2}+\sqrt{2}+\sqrt{\sqrt{\sqrt{2}}}\right)\cdot 2<8$, and let us make the transformations:

$3{,}5<\left(2\sqrt{2}+\sqrt{\sqrt{\sqrt{2}}}\right)<4$,

$3{,}5-2\sqrt{2}<\sqrt{\sqrt{\sqrt{2}}}<4-2\sqrt{2}$,

$\left(3{,}5-2\sqrt{2}\right)^2<\left(\sqrt{\sqrt{\sqrt{2}}}\right)^2<\left(4-2\sqrt{2}\right)^2$,

$12{,}25-14\sqrt{2}+8<\sqrt{\sqrt{2}}<16-16\sqrt{2}+8$,

$\left(20{,}25-14\sqrt{2}\right)^2<\left(\sqrt{\sqrt{2}}\right)^2<\left(24-16\sqrt{2}\right)^2$,

$410{,}0625-567\sqrt{2}+392<\sqrt{2}<576-768\sqrt{2}+512$.

For the inequality in the right side

$769\sqrt{2}<1088$,

$769^2\cdot 2<1088^2$,

$1182722<1183744$, which is correct.

For the inequality in the left side

$802{,}0625<568\sqrt{2}$,

$802{,}0625^2<\left(568\sqrt{2}\right)^2$,

$643304{,}25390625<645248$, which is correct.

Similarly, we can prove that for:

$V=(1,4,4)\Rightarrow 7<A<8$;

$V=(1,4,5)\Rightarrow 6<A<7$;

$V=(2,2,2)\Rightarrow 7<A<8$;

$V=(2,2,3)\Rightarrow 6<A<7$.

Let us examine the expression

$B=(2+2+2)\cdot\underbrace{\sqrt{\ldots\sqrt{2}}}_{v}$.

Analogously to the expression A we can prove that for:

$v=1\Rightarrow 8<A$;

$v = 2 \Rightarrow 7 < A < 8$;

$v \geq 3 \Rightarrow 6 < A < 7$.

Let

$$C = \frac{(2 \cdot 2 \cdot 2)}{\underbrace{\sqrt{\ldots\sqrt{2}}}_{\nu}}.$$

The results from the analysis of C are:

$v = 1 \Rightarrow C < 6$;

$v = 2 \Rightarrow 6 < C < 7$;

$v \geq 3 \Rightarrow 7 < C < 8$.

Depending on the aforementioned assessment of the expressions A, B and C we can choose appropriate operators from the set T_3 (i.e. ceiling or floor functions) for the objective value to be achieved.

The results from the solution of the quiz show that in almost all the options the operators from the set T_3 (i.e. ceiling or floor functions) are necessary. We can give only one solution (11) with a double factorial instead of these operators.

We may avoid the “tricky” use of embedded radicals if we introduce some additional restrictions. For example, if the use of more than one radical is forbidden.

The fact that we will always need an approximation allows us to compose a competition quiz as a modification of the examined one:

Example 2. By using 4 times the digit 2, the four basic arithmetic operations, power, brackets and not more than one radical (root) to get the closest possible value to 7. The winner is the one, who offers the best solution.

The problem can be formalized in the following way:

Determine $\inf|7 - \lambda|$, if

$Q = \{\lambda, 2, 4, S\}$, where:

$$S = \{+\,;\,-\,;\,\cdot\,;\,/\,;\,\wedge;\sqrt{\ }\langle 0...1\rangle\,;[\ ]\langle 0\rangle\,;]\ [\langle 0\rangle\,;!\langle 0\rangle;!!\langle 0\rangle;(\ )\}.$$

Unlike the problem (7), here the parameter $\lambda \in R$.

We will discuss four options:

1) $a = 2 \cdot 2 \cdot 2 - \sqrt{2}$;

2) $b = 2 + 2 + 2 + \sqrt{2}$;

3) $c = \frac{\sqrt{222}}{2}$.

4) $d = \dfrac{2+2}{2-\sqrt{2}}$.

Since the first three options are (8), (9) and (10), we have $6 < a < 7$; $7 < b < 8$ and $7 < c < 8$. It can be shown that $6 < d < 7$ (we leave this to the readers).

We will prove, that $7 - d < 7 - a$.

Both sides of the inequality are positive. We modify the expressions by rationalizing the denominator, canceling and raising to the second power:

$$7 - \frac{2+2}{2-\sqrt{2}} < 7 - \left(2 \cdot 2 \cdot 2 - \sqrt{2}\right),$$

$$7 - \frac{4\left(2+\sqrt{2}\right)}{\left(2-\sqrt{2}\right)\left(2+\sqrt{2}\right)} < 7 - 8 + \sqrt{2},$$

$$7 - \frac{4\left(2+\sqrt{2}\right)}{4-2} < \sqrt{2} - 1,$$

$$7 - \frac{4\left(2+\sqrt{2}\right)}{2} < \sqrt{2} - 1,$$

$$7 - 4 - 2\sqrt{2} < \sqrt{2} - 1,$$

$$4 < 3\sqrt{2},$$

16< 18, which is a correct inequality.

Analogously we can prove that $|7-a| = |7-b| < |7-c|$.

On the real number axis, presented in figure 1, the constants a, b, c, d and 7 are given.

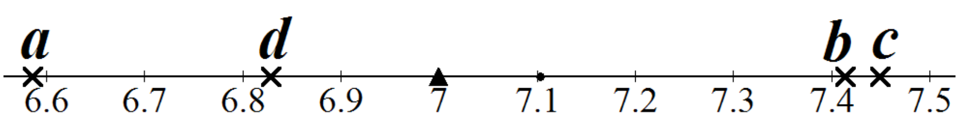

Figure 1

We cannot claim that d is the best possible approximation. In figure 1, close to the mark for 7.1, a point is positioned and its value is closer to 7 in comparison with the aforementioned ones and the conditions in Example 2 are satisfied. We will not reveal the way to achieve this value and we will leave this to the readers who may try to find this solution or even a better one.

During the XVI – XIX centuries the so called "anagram quizzes" were very popular. Those were short permutation cryptograms used to announce some unconfirmed hypotheses in order to respect the scientific priority. We also took a decision to set our priority for determining the best solution (up to now), which was given in figure 1 by an anagram in figure 2. We have done two versions – Bulgarian and English. The anagram is composed by shuffling (permuting) the symbols from a text that describes our "discovery" and the punctuation is skipped. The space symbol is substituted by low line.

If the anagram is decrypted, it will guide you to reach our solution. In addition, the guidance fully corresponds to the spirit of this paper.

TT_SFINWHTQ_T_OEOUTET_N_AW_HP_AROTELR_E_H_UAD_TERSIERH_A_
SGOETDTERONWIW_ET_OXOIE_W_X_T_ORIX

A) English version

ДЦПТВКЗВЯЕ_АОАЕЛНДМРТ_О_ВЕЕЕПТРЕ_НЛЛОА_СН_Ю_ДИ_ИДСНИ_ЧЯА_
АКЗИ_ЖД_ААСПЕВСЛПЛОХЕТ_ИОКХ_ЕОС_АХ

B) Bulgarian version

Figure 2

Both examples that were examined and the partial solutions that were given, propose various ways to formalize and solve problems from different levels of difficulty and from the type of this paper. In both examples some irrational expressions must be examined and, in our opinion, this may improve the comprehension of this particular teaching material at school. Furthermore, such examples introduce some gaming and competition spirit, which may increase the students' interest in mathematics.

NOTES

1. Subfactorial is $!n = n!\left(1+(-1)^1\frac{1}{1!}+(-1)^2\frac{1}{2!}+...+(-1)^k\frac{1}{k!}+...+(-1)^n\frac{1}{n!}\right)$.

2. Bourke, P. (n.d) Four Fours Problem. Retrieved January 22, 2019 from http://paulbourke.net/fun/4444/

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